1. **Definition.**

Let A, B be sets.

(a) *A is said to be* **of cardinality less than or equal to** *B if there is an injective function from A to B.*

We write $A \lesssim B$.

We may also write $B \gtrsim A$ and say B is of cardinality greater than or equal to A.

(b) *A is said to be* **of cardinality less than** *B if (there is an injective function from A to B and there is no bijective function from A to B).*

We write $A < B$.

We may also write $B > A$ and say B is of cardinality greater than A.

Remark on further notations.

- We write $A \not\lesssim B$, or equivalently $B \not\gtrsim A$ exactly when it is not true that $A \lesssim B$,
- We write $A \not< B$, or equivalently $B \not> A$ exactly when it is not true that $A < B$.

2. **Theorem (IX). (Basic properties of $\lesssim$.)**

(0) *Let A, B be sets. Suppose $A \sim B$. Then $A \lesssim B$.*

(1) Let A, B be sets. $A < B$ iff ($A \lesssim B$ and $A \not\sim B$).

(2) *Let A, B be sets. Suppose $A \subset B$. Then $A \lesssim B$.*

(3) Let A be a set. $\emptyset \lesssim A$. If $A \lesssim \emptyset$ then $A = \emptyset$.

(4) *Let A be a set. $A \lesssim A$.*

(5) *Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Then $A \lesssim C$.*

3. **Simple example (1).**

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Then $\mathbb{N} \lesssim \mathbb{Z} \lesssim \mathbb{Q} \lesssim \mathbb{R} \lesssim \mathbb{C}$.

4. **Simple example (2).**

$\mathbb{Q} \lesssim \mathbb{Z}^2$.

Remark. Recall that $\mathbb{Z}^2 \sim \mathbb{N}^2 \sim \mathbb{N}$. Then $\mathbb{Q} \lesssim \mathbb{N}$.

Justification for $\mathbb{Q} \lesssim \mathbb{Z}^2$:

- We take the statement ($\sharp$) for granted:

  ($\sharp$) *For any* $r \in \mathbb{Q}\backslash\{0\}$, *there exist some unique* $p_r, q_r \in \mathbb{Z}$ *such that* $\gcd(p_r, q_r) = 1$ *and* $q_r > 0$ *and* $r = \dfrac{p_r}{q_r}$.

- Define the function $f : \mathbb{Q} \longrightarrow \mathbb{Z}^2$ by

$$f(r) = \begin{cases} (p_r, q_r) & \text{if} \quad r \in \mathbb{Q}\backslash\{0\} \\ (0, 1) & \text{if} \quad r = 0. \end{cases}$$

  f is injective. (Exercise.)

  It follows that $\mathbb{Q} \lesssim \mathbb{Z}^2$.

- Justification for the statement ($\sharp$)? Exercise.

  (Refer to the Handout *Basic results on divisibility* and the Handout *Euclidean Algorithm*.)

5. **Theorem (X). (Further basic properties of $\lesssim$.)**

(1) *Let $f : A \longrightarrow B$ be a function. The following statements hold:*

(1a) *If f is injective then $f(A) \sim A$.*

(1b) $f(A) \lesssim A$.

(2) *Let A, B be non-empty sets.*

$A \lesssim B$ iff (there is a surjective function from B to A.)

(3) *Let A, B, C, D be sets.*

Suppose $A \lesssim C$ and $B \lesssim D$. Then $A \times B \lesssim C \times D$.

(4) *Let A, B be sets.*

Suppose $A \lesssim B$. Then $\mathfrak{P}(A) \lesssim \mathfrak{P}(B)$.

(5) *Let A, B, C, D be non-empty sets.*

Suppose $A \lesssim C$ and $B \lesssim D$. Then $\mathsf{Map}(A, B) \lesssim \mathsf{Map}(C, D)$.

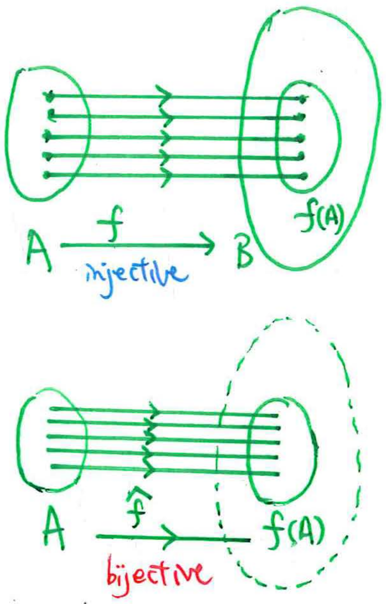
Theorem (X). (Further basic properties of $\lesssim$.)

(1) *Let $f : A \longrightarrow B$ be a function. The following statements hold:*

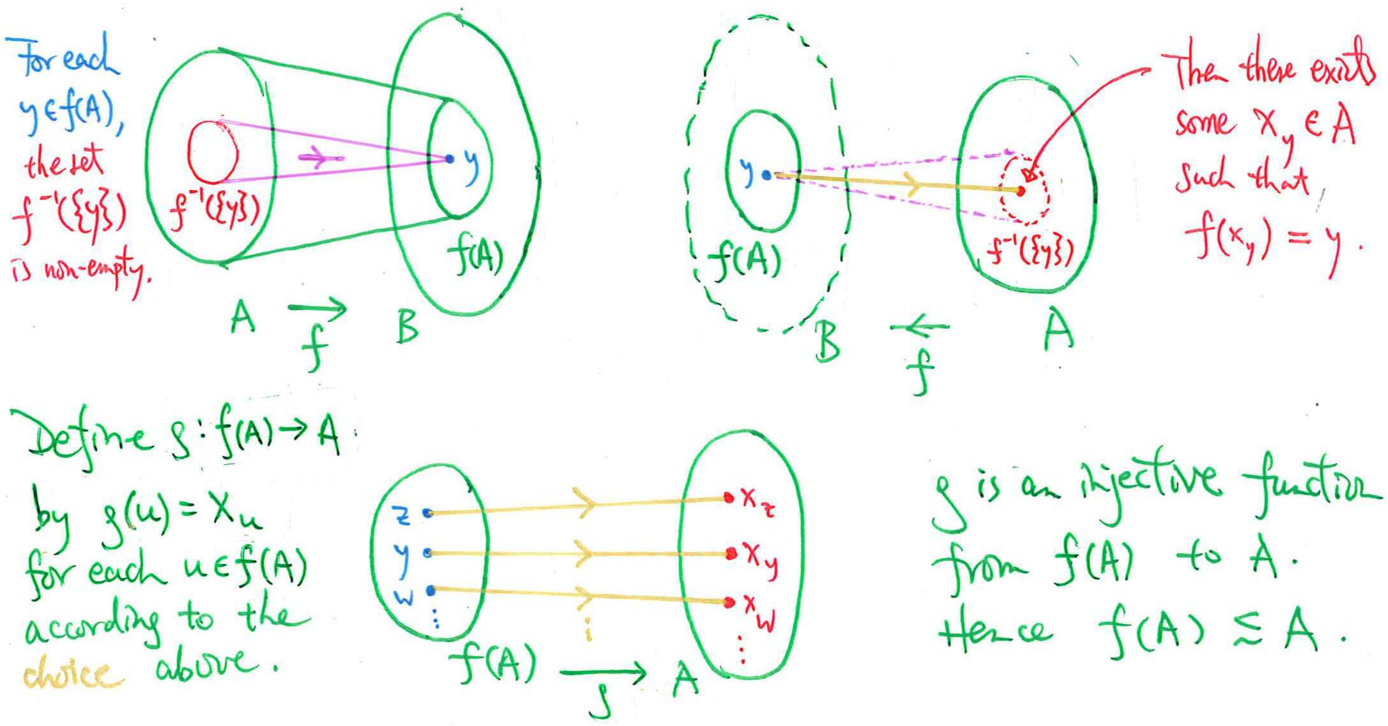
(1a) *If f is injective then $f(A) \sim A$.*

(1b) $f(A) \lesssim A$.

Pictures for (1a):



Pictures for (1b):



6. Two seemingly obvious but non-trivial results about $\lesssim$.

Question.

Consider each of the statements below. Is it true? Or is it false?

(1) '*Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A = B$.* '

(2) '*Let A, B be sets. $A \lesssim B$ or $B \lesssim A$.*

(Exactly one of '$A < B$', '$A = B$', '$A > B$' holds.) '

Why are we bothered with such a question?

Recall that these statements are true:

- '*Let A be a set. $A \lesssim A$.*'
- '*Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Then $A \lesssim C$.*'

It is natural to ask whether

$\lesssim$ 'behaves like' a partial ordering or total ordering,

as the symbol '$\lesssim$' suggests.

Two seemingly obvious but non-trivial results about $\lesssim$.

Question.

Consider each of the statements below. Is it true? Or is it false?

(1) '*Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A = B$.* '

(2) '*Let A, B be sets. $A \lesssim B$ or $B \lesssim A$.*

(Exactly one of '$A < B$', '$A = B$', '$A > B$' holds.) '

Answer.

Statement (1) is false; counter-example?

Statement (1) is false in the sense 'because' it takes too much for a 'set equality' to hold. 'Relaxing' the 'conclusion part' in Statement (1), we do obtain an important true statement.

Theorem (XI). (Schröder-Bernstein Theorem.)

Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A \sim B$.

Two seemingly obvious but non-trivial results about $\lesssim$.

Question.

Consider each of the statements below. Is it true? Or is it false?

(1) '*Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A = B$.* '

(2) '*Let A, B be sets. $A \lesssim B$ or $B \lesssim A$.*

(Exactly one of '$A < B$', '$A = B$', '$A > B$' holds.) '

Answer.

Statement (1) is false; counter-example? Statement (2) is true; proof?

Theorem (XI). (Schröder-Bernstein Theorem.)

Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A \sim B$.

Statement (2) is true and is highly non-trivial; it is a consequence of the Axiom of Choice, under the other 'standard assumptions' of set theory.

Theorem (XII). (Law of Trichotomy.)

Let A, B be sets. $A \lesssim B$ or $B \lesssim A$. (Exactly one of '$A < B$', '$A \sim B$', '$A > B$' holds.)

7. **Axiom of Choice.**

What is this so called Axiom of Choice? There are many (logially equivalent) formulations:

- **(AC1)** *Let* I, M *be non-empty sets, and* $\Phi : I \longrightarrow \mathfrak{P}(M)$ *be a function.*
  Suppose $\Phi(\alpha) \neq \emptyset$ *for any* $\alpha \in I$.
  Then there exists a function $\varphi : I \longrightarrow M$ *such that* $\varphi(\alpha) \in \Phi(\alpha)$ *for any* $\alpha \in I$.
- **(AC2)** *For any non-empty set* A*, there exists some function* $\psi : \mathfrak{P}(A)\backslash\{\emptyset\} \longrightarrow A$ *such that* $\psi(S) \in S$ *for any* $S \in \mathfrak{P}(A)\backslash\{\emptyset\}$.
- **(AC3)** *The cartesian product of any non-empty family of non-empty sets is non-empty.*

Remark.

In the context of the statement (AC1), it is the function φ through which we *choose to assign* each $\alpha \in I$ to the element $\varphi(\alpha)$ of the subset $\Phi(\alpha)$ of M.

In the context of the statement (AC2), the function ψ is called a **choice function**: it is through ψ that we *choose* for each non-empty subset of A an element of the subset concerned.

Why such functions exist in the first place is the problem: intuition suggests they *must* exist, but intuition cannot take the place of reason.

Axiom of Choice.

What is this so called Axiom of Choice? There are many (logially equivalent) formulations:

- **(AC1)** *Let* I, M *be non-empty sets, and* $\Phi : I \longrightarrow \mathfrak{P}(M)$ *be a function.*
  Suppose $\Phi(\alpha) \neq \emptyset$ *for any* $\alpha \in I$.
  Then there exists a function $\varphi : I \longrightarrow M$ *such that* $\varphi(\alpha) \in \Phi(\alpha)$ *for any* $\alpha \in I$.
- **(AC2)** *For any non-empty set* A*, there exists some function* $\psi : \mathfrak{P}(A)\backslash\{\emptyset\} \longrightarrow A$ *such that* $\psi(S) \in S$ *for any* $S \in \mathfrak{P}(A)\backslash\{\emptyset\}$.
- **(AC3)** *The cartesian product of any non-empty family of non-empty sets is non-empty.*

Further remark. Refer to Theorem (X).

(a) We need the Axiom of Choice (the statement (AC1)) in the proof of (1b).

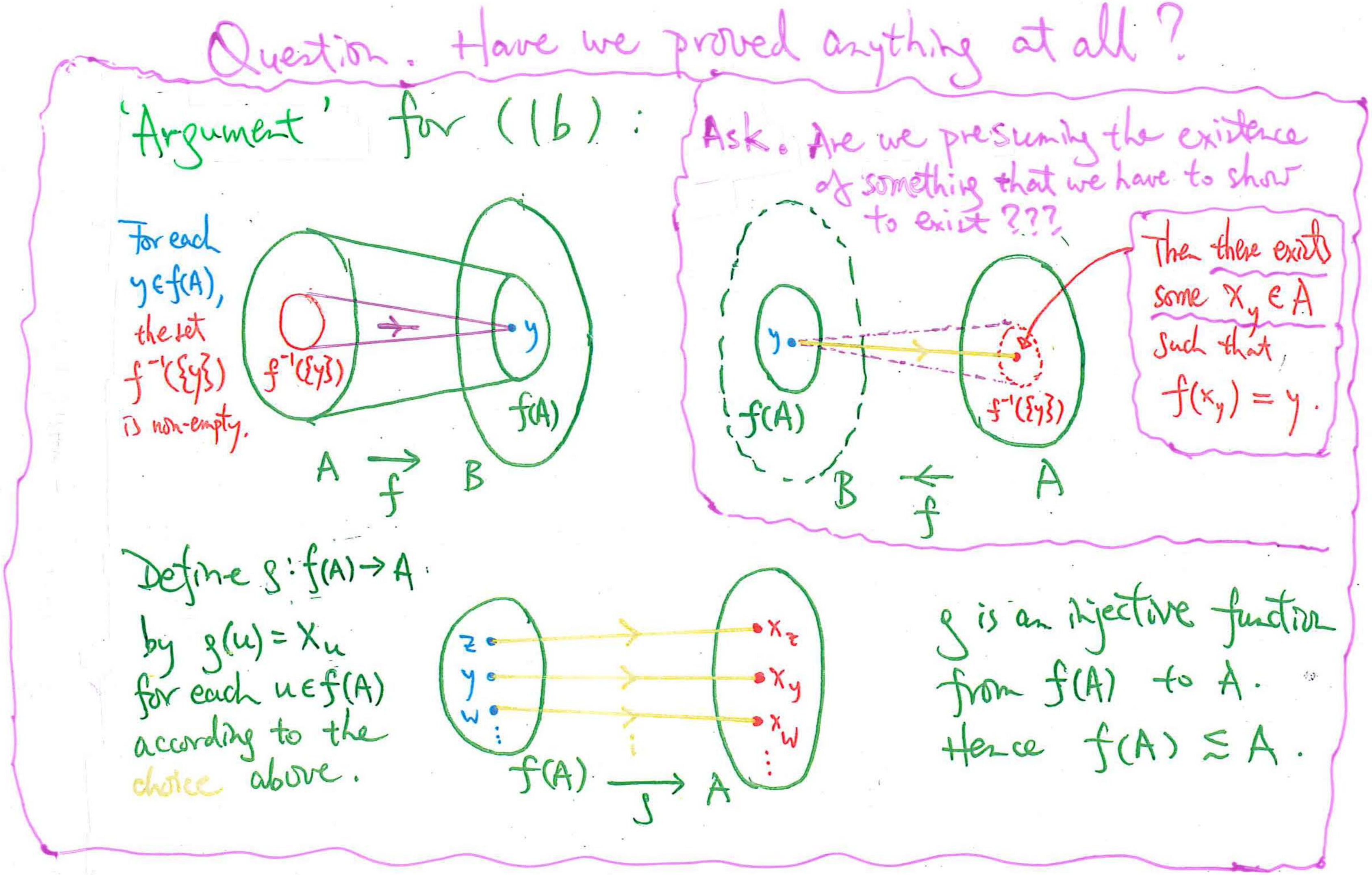
(b) We also need the Axiom of Choice (the statement (AC1)) in the argument for '$\Longleftarrow$-part' of (2).

Theorem (X). (Further basic properties of $\lesssim$.)

(1) *Let $f : A \longrightarrow B$ be a function. The following statements hold:*

(1a) *If f is injective then $f(A) \sim A$.*

(1b) $f(A) \lesssim A$.



Theorem (X). (Further basic properties of $\lesssim$.)

(1) *Let $f : A \longrightarrow B$ be a function. The following statements hold:*

(1a) *If f is injective then $f(A) \sim A$.*

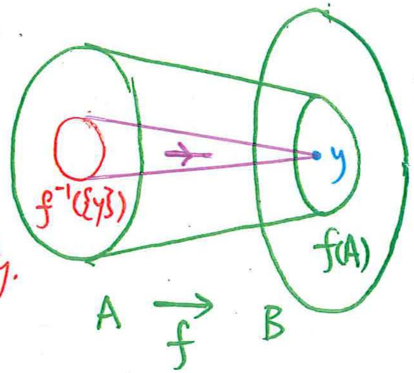
(1b) $f(A) \lesssim A$.

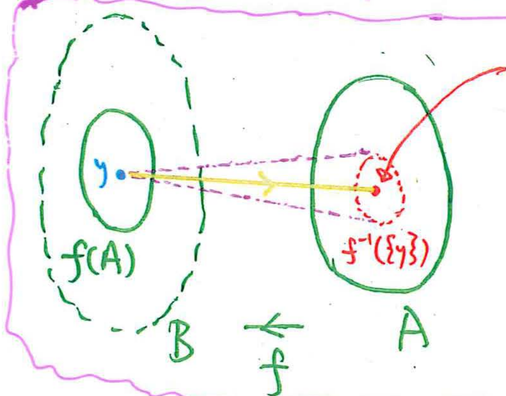
Axiom of Choice.

Let I, M be non-empty sets, and $\Phi : I \to \mathfrak{P}(M)$ be a function.
Suppose $\Phi(\alpha) \neq \emptyset$ for any $\alpha \in I$.
Then there exists a function $\varphi : I \to M$ such that $\varphi(\alpha) \in \Phi(\alpha)$ for any $\alpha \in I$.

'Argument' for (1b): We have used the Axiom of Choice (without being aware of it) here.

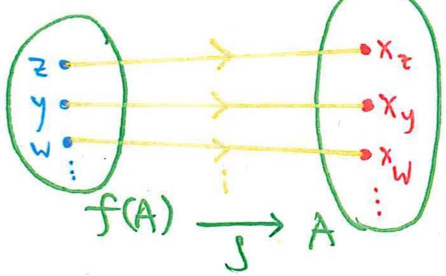
For each $y \in f(A)$, the set $f^{-1}(\{y\})$ is non-empty.





Then there exists some $x_y \in A$ such that $f(x_y) = y$.

Define $g : f(A) \to A$ by $g(u) = x_u$ for each $u \in f(A)$ according to the choice above.



g is an injective function from $f(A)$ to A. Hence $f(A) \lesssim A$.

Theorem (X). (Further basic properties of $\lesssim$.)

(1) *Let $f : A \longrightarrow B$ be a function. The following statements hold:*

(1a) *If f is injective then $f(A) \sim A$.*

(1b) *$f(A) \lesssim A$.*

Suppose $A \neq \emptyset$.

Then:

$I = f(A) \neq \emptyset$.

$M = A \neq \emptyset$.

f defines the function $\Phi : f(A) \longrightarrow \mathfrak{P}(A)$ by

$\Phi(y) = f^{-1}(\{y\})$ for any $y \in f(A)$.

For this function Φ, we have:

$\Phi(y) \neq \emptyset$ for any $y \in f(A)$.

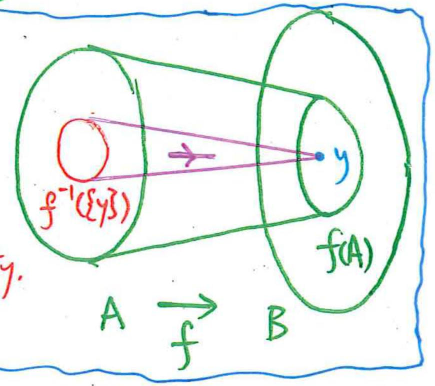
Then, the Axiom of Choice guarantees:

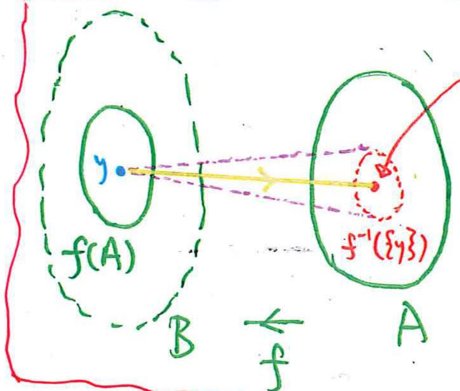
there exists some function $\varphi : f(A) \to A$ such that $\varphi(y) \in f^{-1}(\{y\})$ for any $y \in f(A)$.

This is what we label x_y in the picture.

'Argument' for (1b):

For each $y \in f(A)$, the set $f^{-1}(\{y\})$ is non-empty.





Then there exists some $x_y \in A$ such that $f(x_y) = y$.

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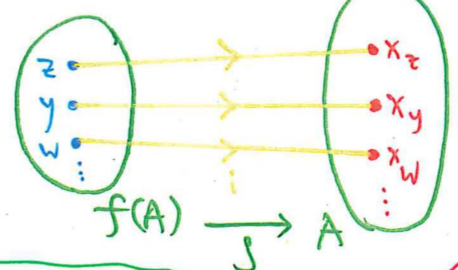

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