1. **Definitions.**

(a) *Let H, K, L be sets.*

The ordered triple (H, K, L) is called a **relation from H to K** *if L is a subset of $H \times K$.*

The set L is called the **graph** *of this relation.*

(b) *Let A, G be sets.*

The ordered triple (A, A, G) is called a **relation in A** *if G is a subset of A^2.*

The set G is called the **graph** *of this relation.*

Remarks.

(1) Every relation in A is a relation from A to A.

(2) Every function from H to K is necessarily a relation from its domain H to its range K.
However, a relation from H to K is not necessarily a function from H to K.

2. **Definitions**.

Let A be a set, and R be a relation in A with graph G.

(a) *R is said to be* **reflexive** *if the statement (ρ) holds:*

(ρ): *For any $x \in A$, $(x, x) \in G$.*

(b) *R is said to be* **symmetric** *if the statement (σ) holds:*

(σ): *For any $x, y \in A$, if $(x, y) \in G$ then $(y, x) \in G$.*

(c) *R is said to be* **transitive** *if the statement (τ) holds:*

(τ): *For any $x, y, z \in A$, if $((x, y) \in G$ and $(y, z) \in G)$ then $(x, z) \in G$.*

Remarks.

(1) The notions of reflexivity, symmetry, and transitivity are 'logically independent' of each other.

(2) How are non-reflexivity, non-symmetry, and non-transitivity formulated? (What are the respective negations of the statements (ρ), (σ), (τ)?)

3. **Definition**.

Let A be a set, and R be a relation in A with graph G.

R is called an **equivalence relation in** *A if R is reflexive, symmetric and transitive.*

4. **Primordial example of equivalence relations: equality (for elements) in a set**.

Let B be a set.

The statements below hold, due to properties of the equality symbol '$=$':

(ρ^*): *For any* $x \in B$, $x = x$.

(σ^*): *For any* $x, y \in B$, *if* $x = y$ *then* $y = x$.

(τ^*): *For any* $x, y, z \in B$, *if* ($x = y$ *and* $y = z$) *then* $x = z$.

Define $G = \{(x, y) \mid x, y \in B \text{ and } x = y\}$.

By definition, for any $x, y \in B$, $x = y$ iff $(x, y) \in G$.

How do the statements (ρ^*), (σ^*), (τ^*) translate?

Primordial example of equivalence relations:
equality (for elements) in a set.

Let B be a set.

The statements below hold, due to properties of the equality symbol '$=$':

(ρ^*): *For any* $x \in B$, $\underbrace{x = x}_{(x,x) \in G}$.

(σ^*): *For any* $x, y \in B$, *if* $\underbrace{x = y}_{(x,y) \in G}$ *then* $\underbrace{y = x}_{(y,x) \in G}$.

(τ^*): *For any* $x, y, z \in B$, *if* ($\underbrace{x = y}_{(x,y) \in G}$ *and* $\underbrace{y = z}_{(y,z) \in G}$) *then* $\underbrace{x = z}_{(x,z) \in G}$.

Define $G = \{(x, y) \mid x, y \in B \text{ and } x = y\}$.

By definition, for any $x, y \in B$, $x = y$ iff $(x, y) \in G$.

Because of $(\rho^*), (\sigma^*), (\tau^*)$, we conclude that (B, B, G) is an equivalence relation.

5. **What motivates the notion of equivalence relations?**

Let R be an equivalence relation in A with graph G.

Suppose we agree to write $x \sim y$ exactly when $(x, y) \in G$.

Then we may think of the equivalence relation R in the set A in a less formal way:

it is some kind of mathematical object represented by the symbol $\sim$, for which the following statements hold simultaneously:

- (ρ): *For any $x \in A$, $x \sim x$.*
- (σ): *For any $x, y \in A$, if $x \sim y$ then $y \sim x$.*
- (τ): *For any $x, y, z \in A$, if ($x \sim y$ and $y \sim z$) then $x \sim z$.*

An equivalence relation R in a set A can be thought of as some **'weaker kind of equality'** for elements of A:

- Even though x, y may be *different* elements of A, we *dis-regard their distinction* 'through the lens' of the equivalence relation R exactly when

$$(x, y) \text{ belongs to graph of } R.$$

6. **Theorem (1).**

Let A, B be sets, and $f : A \longrightarrow B$ be a function.

Define

$$E_f = \{(x, y) \mid x, y \in A \text{ and } f(x) = f(y).\},$$

and $R_f = (A, A, E_f)$.

Then R_f is an equivalence relation in A, with graph E_f.

Remark on terminology.

R_f is called the **equivalence relation in A induced by the function f.**

Further remark.

Through the equivalence relation R_f, we *dis-regard their distinction* between two distinct elements x, y of A exactly when $f(x) = f(y)$.

What is Theorem (1) about?

- Assumption:

$f : A \to B$ is a function.

$E_f = \{(x,y) \mid x, y \in A \text{ and } f(x) = f(y)\}$.

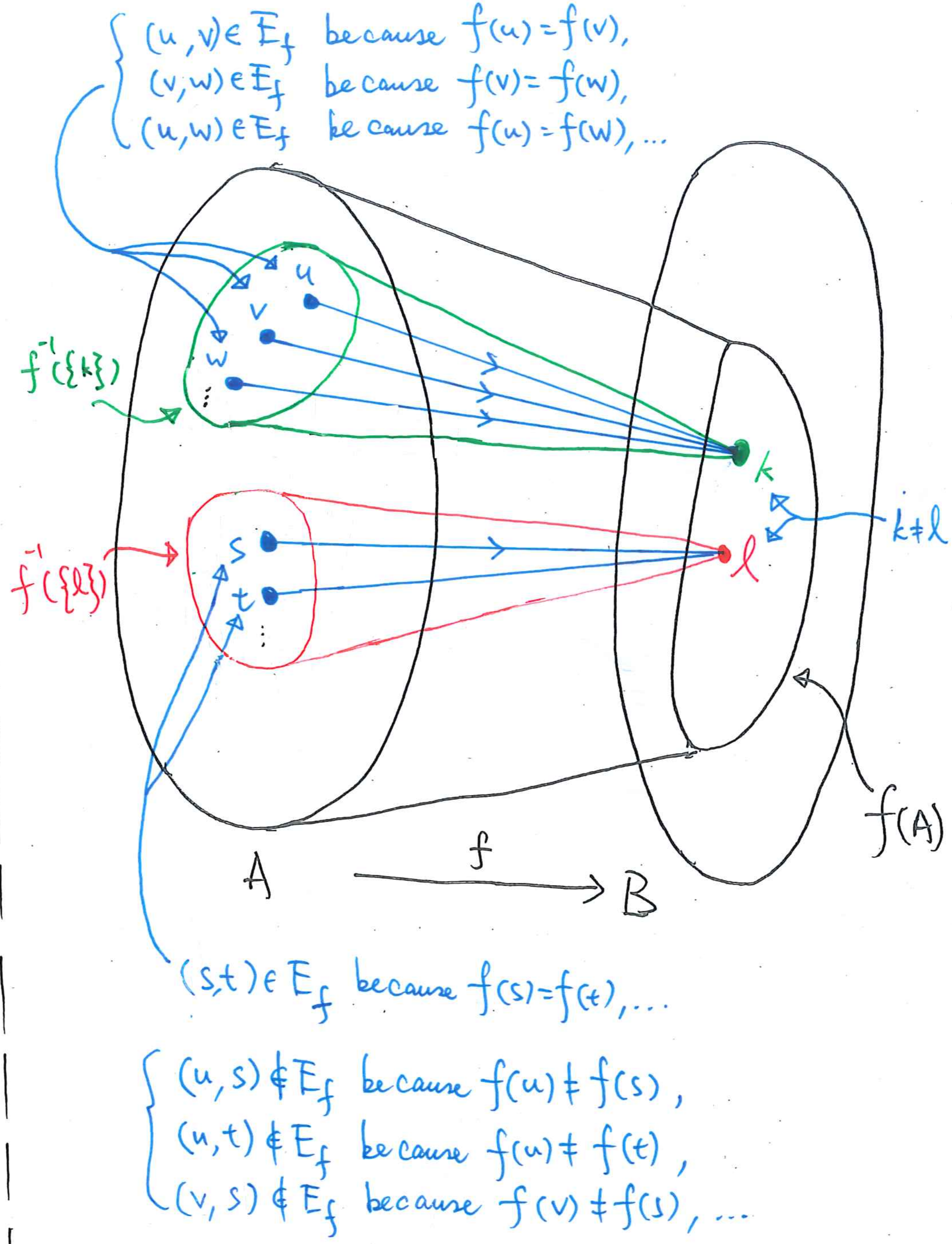
$R_f = (A, A, E_f)$.

So, by definition, for any $x, y \in A$,

$(x,y) \in E_f$ iff $f(x) = f(y)$.

In fact,

$(x,y) \in E_f$ iff x, y belong to the same (non-empty) level set of f.



What is Theorem (1) about?

- Assumption:

  $f : A \to B$ is a function.

  $E_f = \{(x, y) \mid x, y \in A \text{ and } f(x) = f(y)\}$.

  $R_f = (A, A, E_f)$.

- Conclusion:

  R_f is an equivalence relation in A.

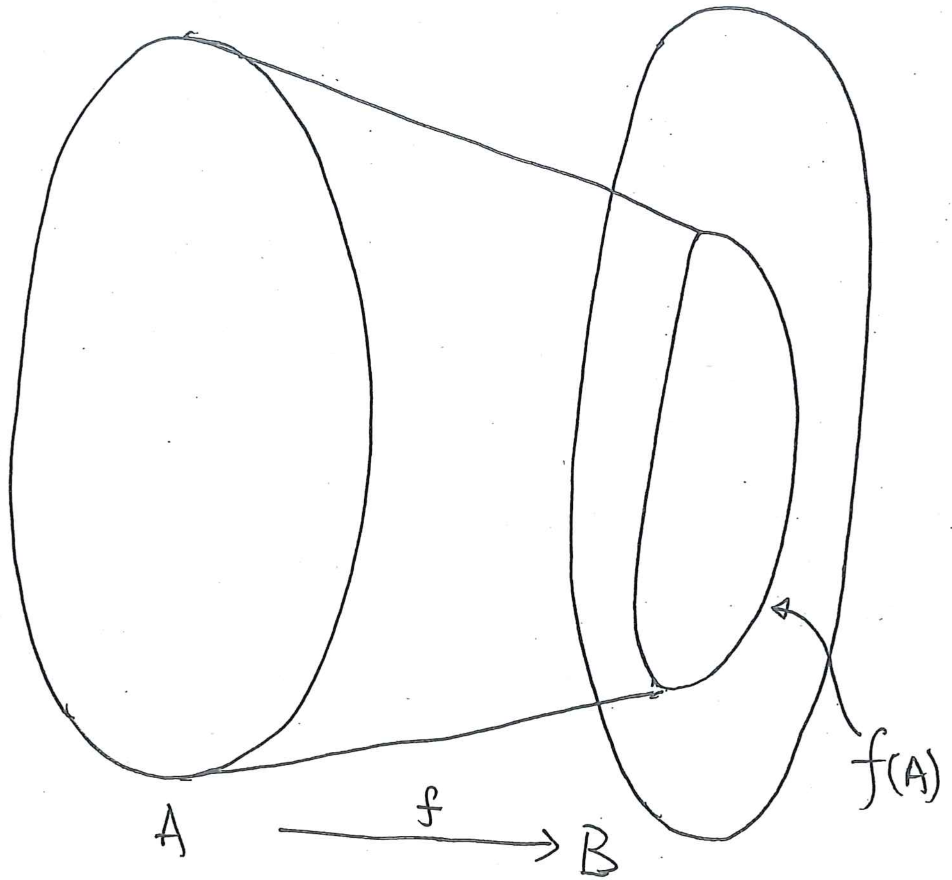
- Reason? This is due to the behaviour of the equality symbol '='.

  The statements (ρ), (σ), (τ) are (trivially) true:

(ρ) For any $u \in A$, $\underbrace{f(u) = f(u)}_{(u,u) \in E_f}$.

(σ) For any $u, v \in A$, if $\underbrace{f(u) = f(v)}_{(u,v) \in E_f}$ then $\underbrace{f(v) = f(u)}_{(v,u) \in E_f}$.

(τ) For any $u, v, w \in A$, if $\underbrace{f(u) = f(v)}_{(u,v) \in E_f}$ and $\underbrace{f(v) = f(w)}_{(v,w) \in E_f}$ then $\underbrace{f(u) = f(w)}_{(u,w) \in E_f}$.



What is Theorem (1) about?

- Assumption:

  $f: A \to B$ is a function.

  $E_f = \{(x,y) \mid x, y \in A \text{ and } f(x) = f(y)\}$.

  $R_f = (A, A, E_f)$.

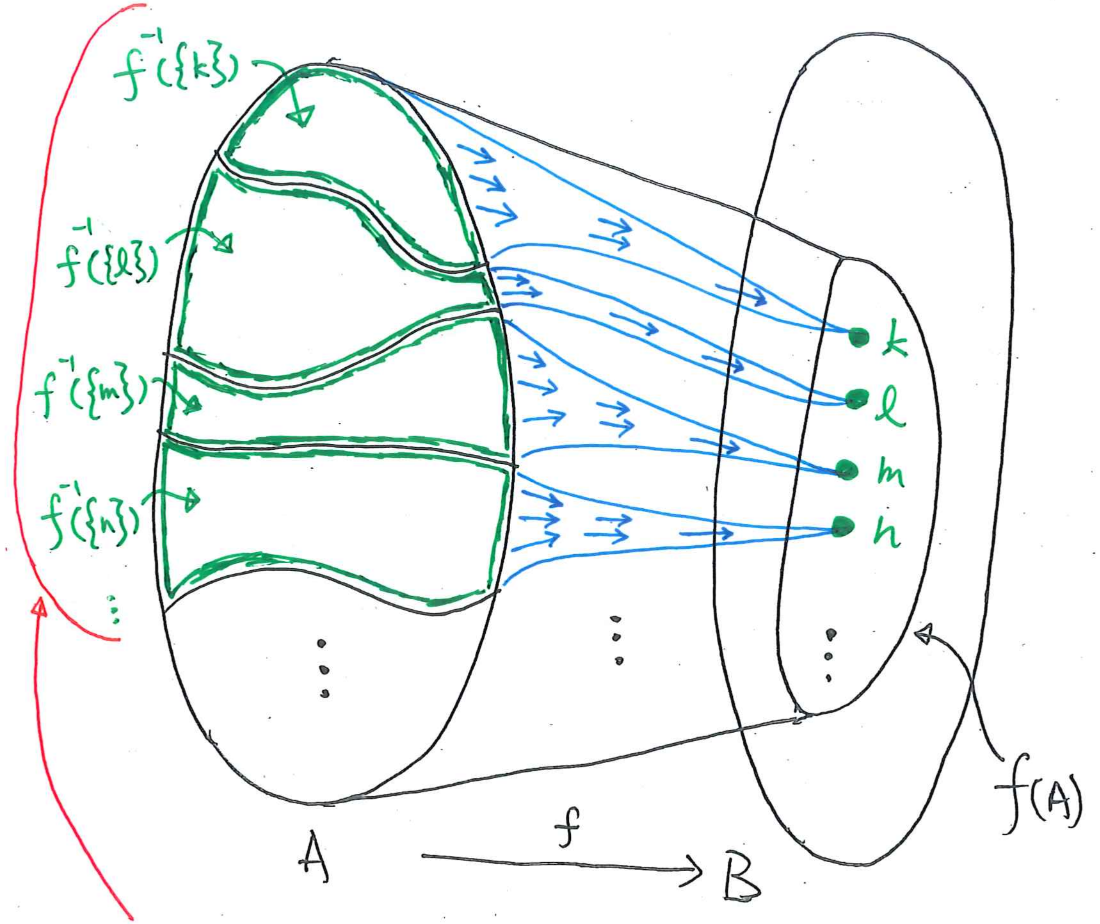
- Conclusion:

  R_f is an equivalence relation in A.

- Consequences of the conclusion:

  Through R_f, we dis-regard (possible) distinction between $x, y \in A$ exactly when $f(x) = f(y)$.

  Then any two elements of A are identified under R_f exactly when they belong to the same (non-empty) level set of f.



Such a collection of subsets of A (here the set of all non-empty level sets of f) is known as a partition of f.

7. **Definition.**

Let A be a set, and Ω be a subset of $\mathfrak{P}(A)$.

Ω is called a **partition of** *A if the statements* (N), (O), (P) *hold:*

(N) *For any $S \in \Omega$, $S \neq \emptyset$.*

(O) $\{x \in A : x \in S$ *for some* $S \in \Omega\} = A$.

(P) *For any $S, T \in \Omega$, exactly one of the statements* (P1), (P2) *holds:*

(P1) $S = T$. (P2) $S \cap T = \emptyset$.

Remarks.

(a) The set $\{x \in A : x \in S$ for some $S \in \Omega\}$ is called the **generalized union** of the set Ω of subsets of A, and is often denoted by $\bigcup_{S \in \Omega} S$.

The statement (O) can be re-written as '$\bigcup_{S \in \Omega} S = A$'.

(b) Two sets K, L are said to be **disjoint** if $K \cap L = \emptyset$.

The statement (P) can be re-written as '*The elements of Ω are pairwise disjoint subsets of A*'.

8. **Theorem (2).**

Let A be a set, and Ω be a partition of A.

Define

$$E_\Omega = \left\{ (x, y) \;\middle|\; \begin{array}{l} x, y \in A, \text{ and there exist some } S \in \Omega \\ \text{such that } x \in S \text{ and } y \in S. \end{array} \right\},$$

and $R_\Omega = (A, A, E_\Omega)$.

Then R_Ω is an equivalence relation in A, with graph E_Ω.

Remark on terminology.

R_Ω is called the **equivalence relation in A induced by the partition Ω.**

What is Theorem (2) about?

- Assumption:

  Ω is a partition of A.

  $E_\Omega = \left\{ (x,y) \;\middle|\; \begin{array}{l} x, y \in A \text{ and} \\ \text{there exists some } S \in \Omega \\ \text{such that } x \in S \text{ and } y \in S \end{array} \right\}$.

  $R_\Omega = (A, A, E_\Omega)$.

  So, by definition of Ω as a partition of A:

(N) For any $S \in \Omega$, $S \neq \phi$.

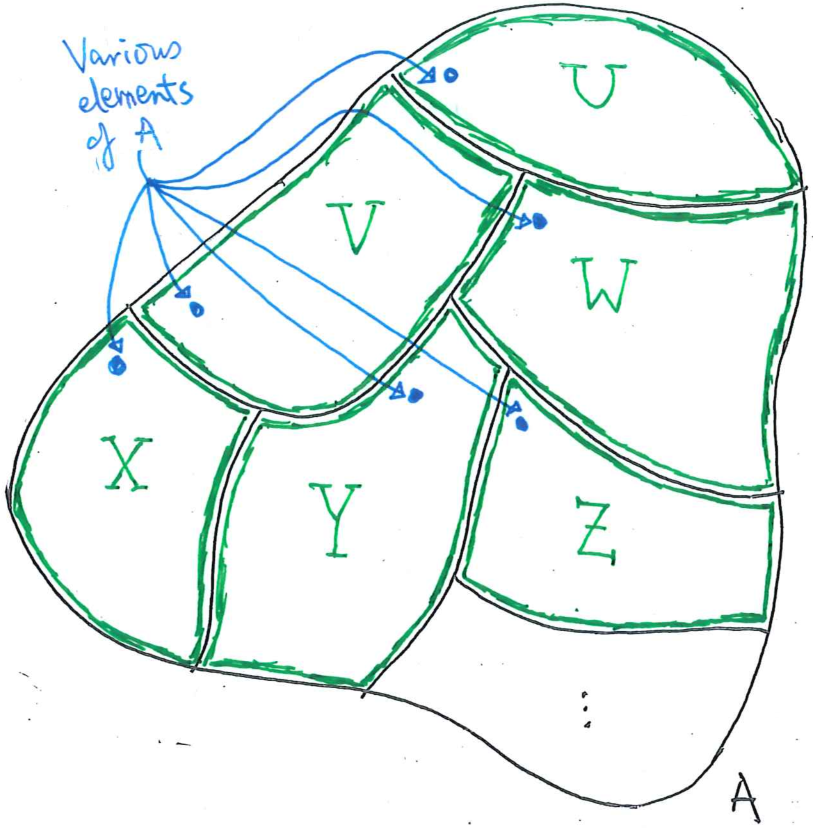
(O) $\{x \in A : x \in S \text{ for some } S \in \Omega\} = A$

(This is called the generalized union of Ω.)

(P) For any $S, T \in \Omega$, exactly one of '$S = T$', '$S \cap T = \phi$' is true.

(S, T are 'disjoint'.)

$\Omega = \{U, V, W, X, Y, Z, \ldots\}$



(N) Each of U, V, W, X, Y, Z is non-empty.

(O) '$U \cup V \cup W \cup X \cup Y \cup Z \cup \ldots$' $= A$.

(P) $U \cap V = \phi$, $U \cap W = \phi$, $U \cap X = \phi$, ...;

$V \cap W = \phi$, $V \cap X = \phi$, ...;

$W \cap X = \phi$, ...; ... .

What is Theorem (2) about?

- Assumption:

  Ω is a partition of A.

  $E_\Omega = \{(x,y) \mid x, y \in A$ and there exists some $S \in \Omega$ such that $x \in S$ and $y \in S\}$.

  $R_\Omega = (A, A, E_\Omega)$.

  So, by definition, for any $x, y \in A$,

  $(x,y) \in E_\Omega$ iff there exists some $S \in \Omega$ such that $x \in S$ and $y \in S$.

  Or simply,

  $(x,y) \in E_\Omega$ iff x, y belong to the same (non-empty) subset of A 'collected' as some element of Ω.

- Conclusion:

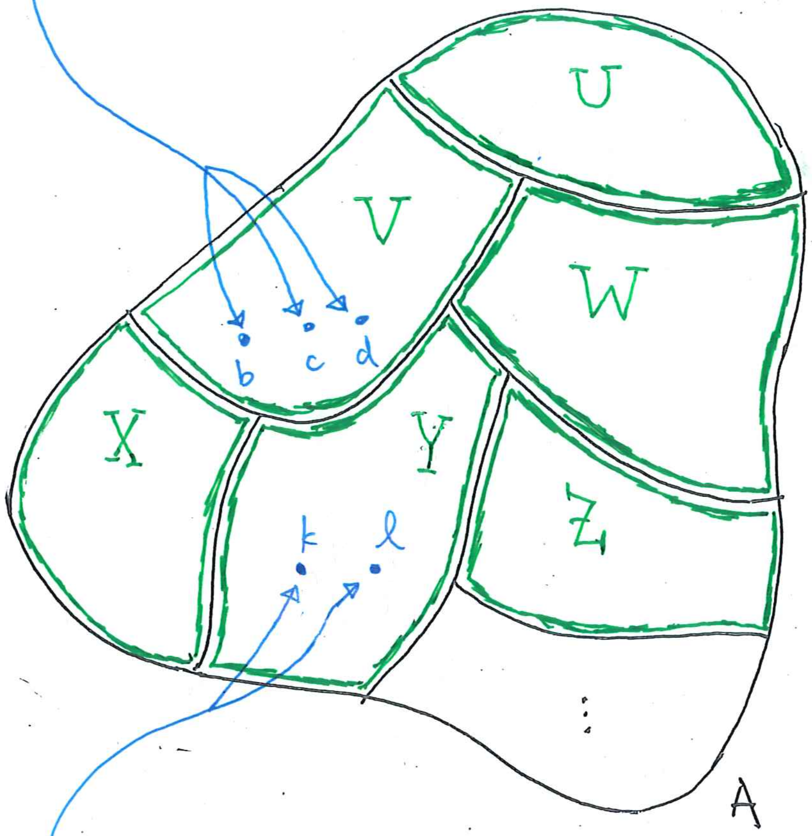
  R_Ω is an equivalence relation in A.

  (Proof? Exercise in 'definition of partition' and 'there exist'.)

$(b,c) \in E_\Omega$ because $b \in V$, $c \in V$ and $V \in \Omega$

$(c,d) \in E_\Omega$ because $c \in V$, $d \in V$ and $V \in \Omega$

$(b,d) \in E_\Omega$ because $b \in V$, $d \in V$ and $V \in \Omega$



$(k,l) \in E_\Omega$ because $k \in Y$, $l \in Y$ and $Y \in \Omega$.

$(b,k) \notin E_\Omega$ because $b \in V$, $k \in Y$, $V \in \Omega$, $Y \in \Omega$ and $V \neq Y$.

$(c,k) \notin E_\Omega$ because $c \in V$, $k \in Y$, $V \in \Omega$, $Y \in \Omega$ and $V \neq Y$.

$(b,l) \notin E_\Omega$ because $b \in V$, $l \in Y$, $V \in \Omega$, $Y \in \Omega$ and $V \neq Y$.

What is Theorem (2) about?

- Assumption:

  Ω is a partition of A.

  $E_\Omega = \left\{ (x, y) \;\middle|\; \begin{array}{l} x, y \in A \text{ and} \\ \text{there exists some } S \in \Omega \\ \text{such that } x \in S \text{ and } y \in S. \end{array} \right\}$.

  $R_\Omega = (A, A, E_\Omega)$.

- Conclusion:

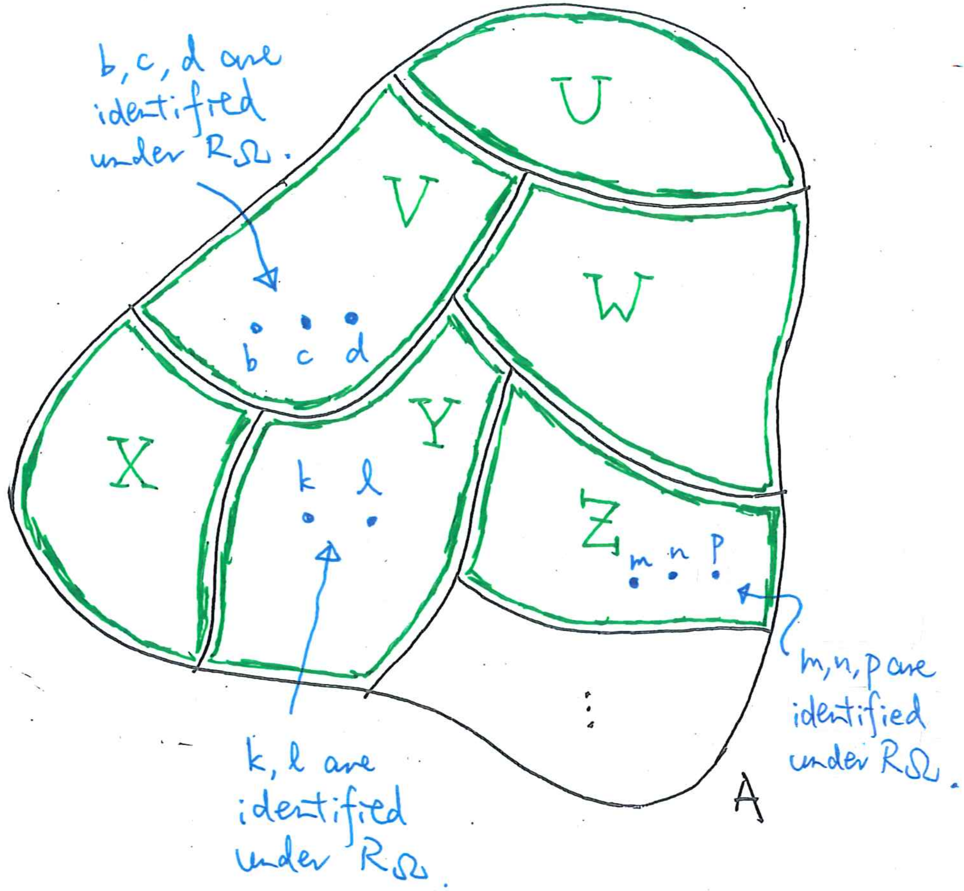
  R_Ω is an equivalence relation in A.

- Consequences of the conclusion:

  Through R_Ω, we dis-regard (possible) distinction between $x, y \in A$ exactly when they belong to the same element of Ω.

  Then any two elements of A are identified under R_Ω exactly when they belong to the same (non-empty) subset of A 'collected' as an element of Ω.

$\Omega = \{U, V, W, X, Y, Z, \ldots\}$



9. **Theorem (3).**

Let A be a set, and R be an equivalence relation in A with graph E.

For any $x \in A$, define $R[x] = \{y \in A : (x, y) \in E\}$.

Define $\Omega_R = \{S \in \mathfrak{P}(A) : S = R[x]$ for some $x \in A\}$.

Define $q_R : A \longrightarrow \Omega_R$ by $q_R(x) = R[x]$ for any $x \in A$.

Then the statements below hold:

(a) *Ω_R is a partition of A.*

(b) *q_R is a surjective function.*

(c) *The equivalence relation R_{Ω_R} in A induced by the partition Ω_R is R itself.*
The equivalence relation R_{q_R} in A induced by the function q_R is R itself.

Remark on terminology.

Note that Ω_R is a special partition of A induced by the equivalence relation R, and q_R is a special surjective function with domain A induced by the equivalence relation R.

(a) For any $x \in A$, the set $R[x]$ is called the **equivalence class of x under the equivalence relation R**.

(b) Ω_R is called the **quotient in A by the equivalence relation R**, and is denoted by A/R.

(c) q_R is called the **quotient mapping of the equivalence relation R**.

Further remark.

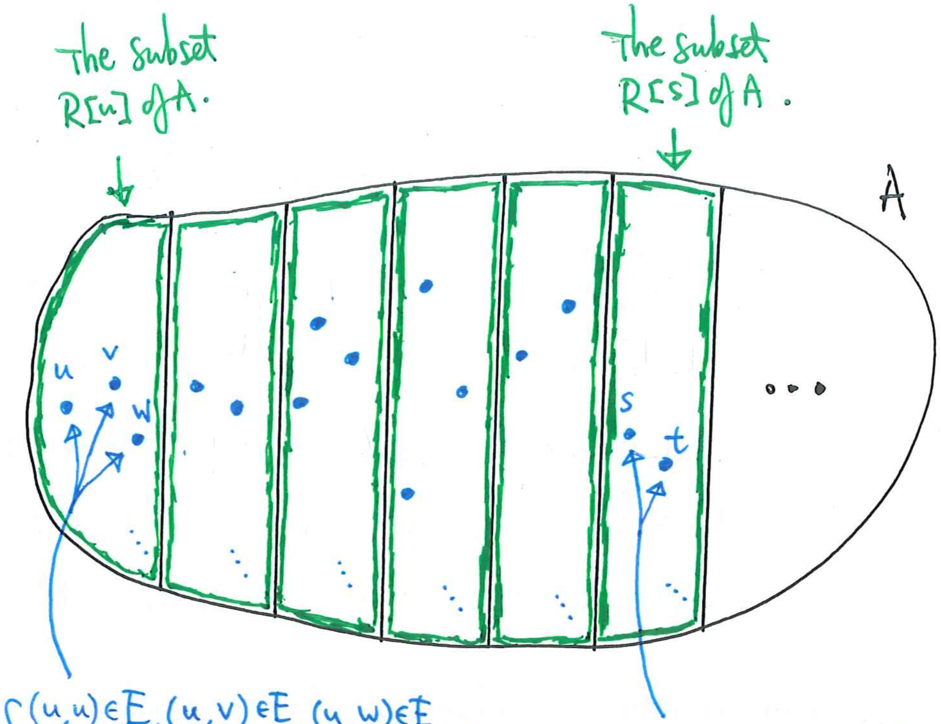
An equivalence relation can be visualized through its quotient and its quotient mapping, in the sense that the information about the equivalence relation is carried in full in both its quotient and its quotient mapping. This is the point of the equalities '$R_{A/R} = R = R_{q_R}$'.

What is Theorem (3) about?

- Assumption:

  R is an equivalence relation in A with graph E.

  For any $x \in A$, $R[x] = \{y \in A : (x,y) \in E\}$.



$(u,u) \in E, (u,v) \in E, (u,w) \in E, \ldots$
$(v,u) \in E, (v,v) \in E, (v,w) \in E, \ldots$
$(w,u) \in E, (w,v) \in E, (w,w) \in E, \ldots$
$\vdots$

Then

$R[u] = R[v] = R[w] = \{u, v, w, \ldots\}$.

$(s,s) \in E, (s,t) \in E, \ldots$
$(t,s) \in E, (t,t) \in E, \ldots$
$\vdots$

Then

$R[s] = R[t] = \{s, t, \ldots\}$.

What is Theorem (3) about?

- Assumption:

  R is an equivalence relation in A with graph E.

  For any $x \in A$, $R[x] = \{y \in A : (x, y) \in E.\}$.

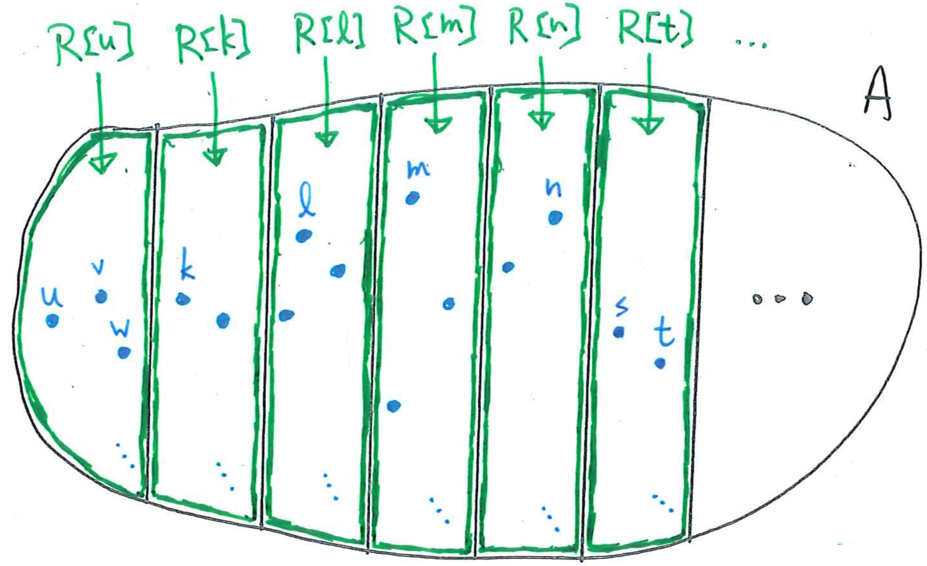
  $\Omega_R = \{S \in \mathfrak{P}(A) : S = R[x] \text{ for some } x \in A.\}$.

- Conclusion:

  Ω_R is a partition of A.

  The equivalence relation in A induced by the partition Ω_R in A is R itself.

  (Proof? Exercise in playing with definitions.)



Ω_R '=' $\{R[u], R[k], R[l], R[m], R[n], R[t], \ldots\}$

Why is such an Ω_R a partition of A?

- For any $S \in \Omega_R$, $S \neq \emptyset$. Why?

  $u \in R[u]$ because $(u, u) \in E$;

- $\{x \in A : x \in S \text{ for some } S \in \Omega_R\} = A$. Why?

  $u \in R[u]$ and $R[u] \in \Omega_R$, $v \in R[v]$ and $R[v] \in \Omega_R$, ...;

  $k \in R[k]$ and $R[k] \in \Omega_R$, ...; ...

- For any $S, T \in \Omega_R$, exactly one of '$S = T$', '$S \cap T = \emptyset$' is true. Why?

  $(u, t) \notin E$ gives 'disjoint-ness' of $R[u], R[t]$;

  $(u, v) \in E$ gives 'identity' of $R[u], R[v]$; ...

What is Theorem (3) about?

- Assumption:

R is an equivalence relation in A with graph E.

For any $x \in A$, $R[x] = \{y \in A : (x, y) \in E.\}$.

$\Omega_R = \{S \in \mathfrak{P}(A) : S = R[x] \text{ for some } x \in A.\}$.

$q_R : A \to \Omega_R$ is the function given by

$q_R(x) = R[x]$ for any $x \in A$.

- Conclusion:

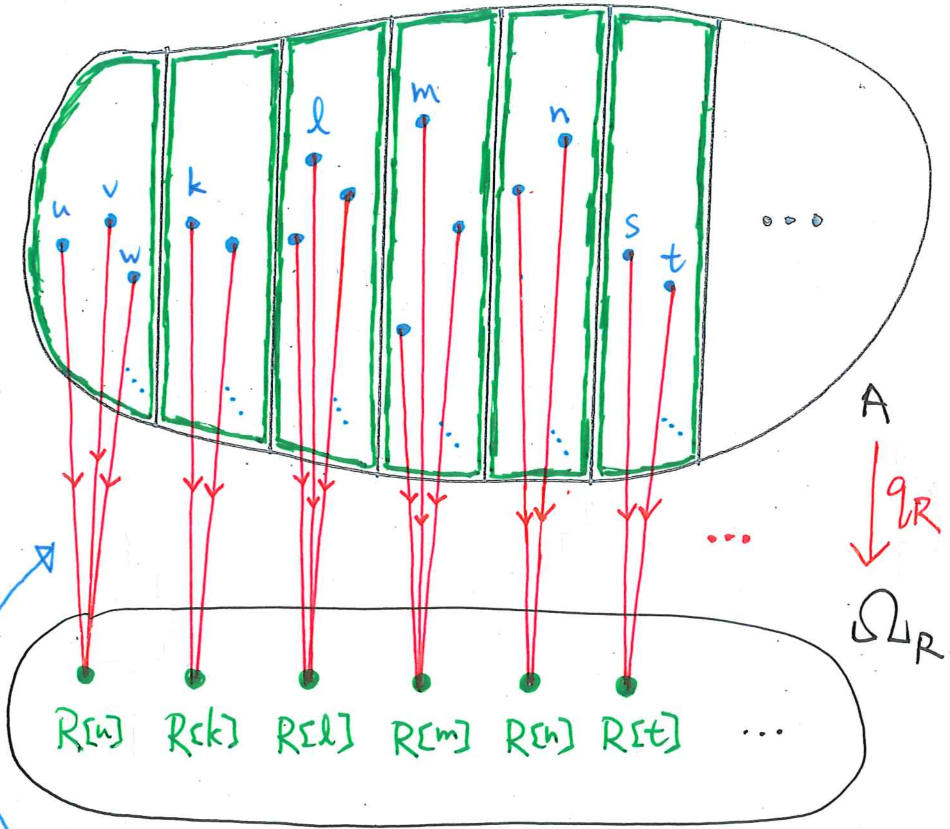
Ω_R is a partition of A.

The equivalence relation in A induced by the partition Ω_R in A is R itself.

q_R is a surjective function.

The equivalence relation in A induced by the function q_R in A is R itself.

(Proof? Exercise in playing with definitions.)



$q_R(u) = q_R(v) = q_R(w) = \ldots = R[u]$
$= R[v]$
$= R[w] = \ldots$

$q_R(s) = q_R(t) = \ldots = R[t]$
$= R[s] = \ldots$

Why is q_R surjective?

Every $S \in \Omega_R$ is $R[x]$ for some $x \in A$.

Then for this $x \in A$, $q_R[x] = R[x] = S$.

What is Theorem (3) about?

- Assumption:

  R is an equivalence relation in A with graph E.

  For any $x \in A$, $R[x] = \{ y \in A : (x, y) \in E \}$.

  $\Omega_R = \{ S \in \mathcal{P}(A) : S = R[x] \text{ for some } x \in A \}$.

  $q_R : A \to \Omega_R$ is the function given by $q_R(x) = R[x]$ for any $x \in A$.

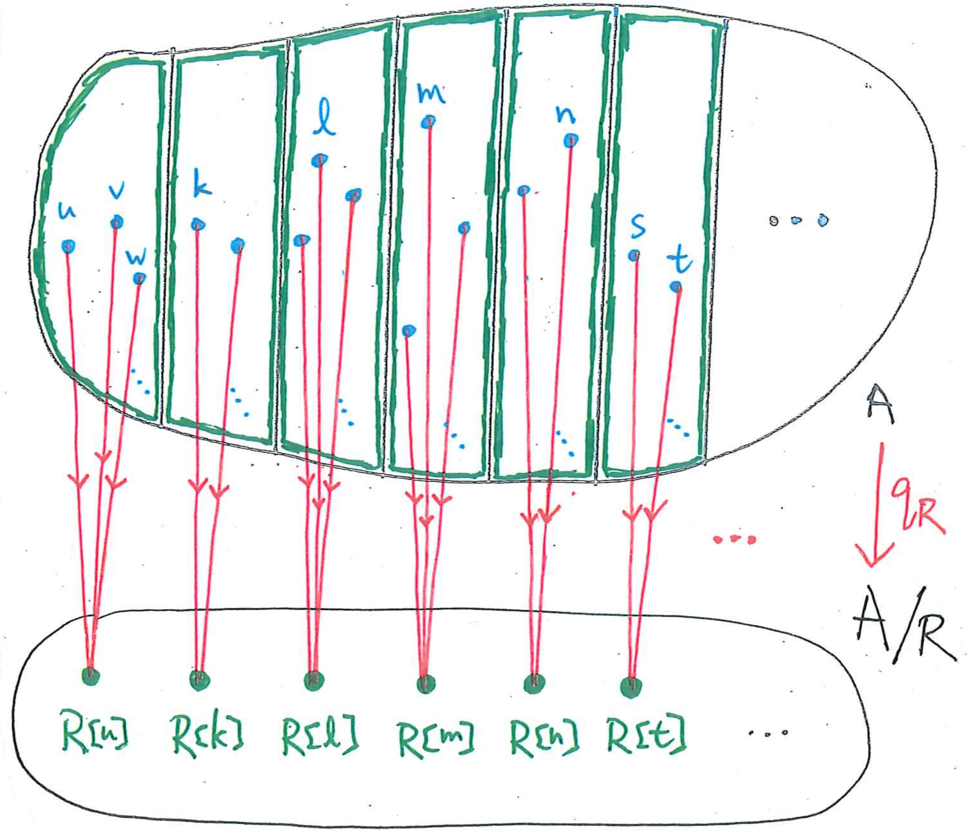
- Conclusion:

  Ω_R is a partition of A.

  The equivalence relation in A induced by the partition Ω_R in A is R itself.

  q_R is a surjective function.

  The equivalence relation in A induced by the function q_R in A is R itself.



Common terminologies and notations.

$R[x]$: Equivalence class of x under R.

Ω_R : Quotient of A by R, usually written as A/R.

q_R : Quotient mapping of R.