1. **Formal definition for the notion of relations and functions, and re-formulation of the definition of surjectivity and injectivity.**

   (a) *Let D, R, H be sets. The ordered triple (D, R, H) is called a* **relation from** D **to** R *if $H \subset D \times R$.*

   (b) *Let D, R be sets, and H be a subset of $D \times R$. The relation (D, R, H) is said to be a* **function from domain** D **to range** R **with graph** H *if both of the statements $(E), (U)$ below hold:*

   (E): *For any $t \in D$, there exists some $u \in R$ such that $(t, u) \in H$.*
   (U): *For any $t \in D$, for any $u, v \in R$, if $(t, u) \in H$ and $(t, v) \in H$ then $u = v$.*

   Where we refer to (D, R, H) as h, we write $u = h(t)$ exactly when $(t, u) \in H$.

   (c) *Let D, R be sets, and $h : D \longrightarrow R$ be a function from D to R with graph H.*

   i. *h is said to be* **surjective** *if the statement (S) below holds:*
   (S): *For any $u \in R$, there exists some $t \in D$ such that $(t, u) \in H$.*

   ii. *h is said to be* **injective** *if the statement (I) below holds:*
   (I): *For any $u \in R$, for any $t, s \in D$, if $(t, u) \in H$ and $(s, u) \in H$ then $t = s$.*

2. **Definition for the notion of bijective function.**

   Let D, R be sets, and $h : D \longrightarrow R$ be a function from D to R. h is said to be **bijective** *if h is both surjective and injective.*

   Remark. Hence $h = (D, R, H)$ is a bijective function from D to R with graph H iff all of the statements $(E), (U), (S), (I)$ below hold:

   (E): *For any $t \in D$, there exists some $u \in R$ such that $(t, u) \in H$.*
   (U): *For any $t \in D$, for any $u, v \in R$, if $(t, u) \in H$ and $(t, v) \in H$ then $u = v$.*
   (S): *For any $u \in R$, there exists some $t \in D$ such that $(t, u) \in H$.*
   (I): *For any $u \in R$, for any $t, s \in D$, if $(t, u) \in H$ and $(s, u) \in H$ then $t = s$.*

3. **Definition. (Notion of inverse function).**

   Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions. g is said to be an **inverse function** *of f if both of the following statements hold:*

   (†) *For any $x \in A$, $(g \circ f)(x) = x$.*
   (‡) *For any $y \in B$, $(f \circ g)(y) = y$.*

   Definition. (Identity function.)

   Let C be a set. Define the function $\mathsf{id}_C : C \longrightarrow C$ by $\mathsf{id}_C(z) = z$ for any $z \in C$. id_C is called the **identity function** *on the set C.*

   Theorem (1). (Re-formulation of the definition of inverse function.)

   Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions. The statements below are logically equivalent:

   $(\star_0)$ *g is an inverse function of f.*
   $(\star_1)$ *$g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$ as functions.*
   $(\star_2)$ *f is an inverse function of g.*
   $(\star_3)$ *For any $x \in A$, for any $y \in B$, ($y = f(x)$ iff $x = g(y)$).*

4. **Theorem (2). (Uniqueness of inverse function.)**

   Let A, B be sets, and $f : A \longrightarrow B$ be a function. f has at most one inverse function.

   Theorem (3). (Necessary condition for existence of inverse function.)

   Let A, B be sets, $f : A \longrightarrow B$ be a function. Suppose f has an inverse function, say, $g : B \longrightarrow A$. Then each of f, g is bijective.

   Question. *Is the necessary condition sufficient as well? Why?*

   Answer. *Yes. Reason: Theorem (4).*

5. **Theorem (4). (Existence and Uniqueness of inverse function for a bijective function.)**

   Let A, B be sets, and $f : A \longrightarrow B$ be a function. Suppose f is bijective. Then there exists some unique bijective function $g : B \longrightarrow A$ such that g is the inverse function of f.

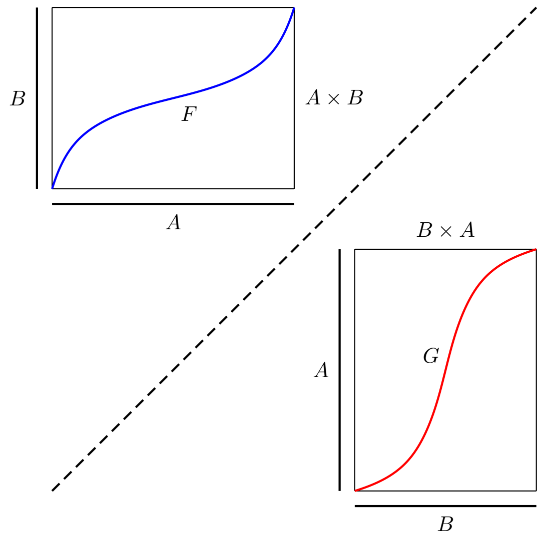
   Convention on notations. Because of the uniqueness of g as the inverse function of a function f (when such exists), we agree to write g as f^{-1}.

6. **Proof of Theorem (4).**

   Let A, B be sets, and $f : A \longrightarrow B$ be a function. Suppose f is bijective.

   [Ask: How to write down an inverse function of f? What will its graph be?]

   Denote by F the graph of f. Define $G = \{(y, x) \mid x \in A \text{ and } y \in B \text{ and } (x, y) \in F\}$.



   By definition, $G \subset B \times A$. Moreover, the statement $(\sharp)$ holds:

   $(\sharp)$: For any $t \in A$, for any $u \in B$, $((t, u) \in F$ iff $(u, t) \in G)$.

   Define g to be the ordered triple (B, A, G). By definition, g is a relation from B to A with graph G. [We verify that g is a bijective function and g is an inverse function of f.]

   Since $f : A \longrightarrow B$ is a bijective function, the following statements hold:

   (E): For any $x \in A$, there exists some $y \in B$ such that $(x, y) \in F$.
   (U): For any $x \in A$, for any $y, z \in B$, if $(x, y) \in F$ and $(x, z) \in F$ then $y = z$.
   (S): For any $y \in B$, there exists some $x \in A$ such that $(x, y) \in F$.
   (I): For any $y \in B$, for any $x, w \in A$, if $(x, y) \in F$ and $(w, y) \in F$ then $x = w$.

   Consider the relation $g = (B, A, G)$. [We are going to apply $(\sharp)$.]

   By (S), the statement (E') holds: for any $y \in B$, there exists some $x \in A$ such that $(y, x) \in G$.
   By (I), the statement (U') holds: for any $y \in B$, for any $x, w \in A$, if $(y, x) \in G$ and $(y, w) \in G$ then $x = w$.
   Therefore g is a function from B to A.
   By (E), the statement (S') holds: for any $x \in A$, there exists some $y \in B$ such that $(y, x) \in G$.
   Hence g is a surjective function.
   By (U), the statement (I') holds: for any $x \in A$, for any $y, z \in B$, if $(y, x) \in G$ and $(z, x) \in G$ then $y = z$.
   Hence g is an injective function. It follows that g is a bijective function from B to A.

   [Ask: Is g indeed an inverse function of f?]

   Pick any $x \in A$, $y \in B$. Note that $y = f(x)$ iff $(x, y) \in F$ iff $(y, x) \in G$ iff $x = g(y)$. It follows from Theorem (1) that g is an inverse function of f. By Theorem (2), g is the unique inverse function of f.

7. **Theorem (5).**

   Let A, B, C be sets and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be bijective functions.

   The statements below hold:

(a) $f^{-1} \circ f = \mathsf{id}_A$ *and* $f \circ f^{-1} = \mathsf{id}_B$.

(b) *For any* $x \in A$, *for any* $y \in B$, $y = f(x)$ *iff* $x = f^{-1}(y)$.

(c) f^{-1} *is a bijective function. Moreover,* $(f^{-1})^{-1} = f$.

(d) $g \circ f$ *is a bijective function. Moreover,* $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Remark. The proof of Theorem (5) is left as an exercise.