

MATH1050 Characterization of surjectivity with image sets, pre-image sets

1. The statement (♯) is true:

(♯) Let A, B be sets and $f : A \rightarrow B$ be a function. For any subset U of B , $f(f^{-1}(U)) \subset U$.

Proof of the statement (♯)?

• Let A, B be sets and $f : A \rightarrow B$ be a function. Let U be a subset of B .

* Pick any object y . Suppose $y \in f(f^{-1}(U))$. Then there exists some $x \in f^{-1}(U)$ such that $y = f(x)$. Since $x \in f^{-1}(U)$, there exists some $z \in U$ such that $z = f(x)$. Now $y = f(x) = z \in U$.

It follows that $f(f^{-1}(U)) \subset U$.

2. The statement (b) is false:

(b) Let A, B be sets and $f : A \rightarrow B$ be a function. For any subset U of B , $f(f^{-1}(U)) \supset U$.

Dis-proof of the statement (b)?

• Take $A = \{0\}$, $B = \{1, 2\}$, $U = \{1, 2\}$. $U \subset B$. Define the function $f : A \rightarrow B$ by $f(0) = 1$. $f^{-1}(U) = \{0\}$. $f(f^{-1}(U)) = \{1\}$. We have $2 \in U$ and $2 \notin f(f^{-1}(U))$. Then $U \not\subset f(f^{-1}(U))$.

3. Follow-up questions:

(a) Can we impose further assumption on f to make the conclusion in the statement (b) hold? (We are looking for some sufficient condition(s) for the statement (b).)

To conceive such an assumption out of nothing may be difficult. Hence we postpone this question.

(b) What must happen to f if the conclusion in the statement (b) is true? (We are looking for some necessary condition(s) for the statement (b).)

• Suppose that for any subset U of B , $f(f^{-1}(U)) \supset U$. [So what happens?]

Note that B is a subset of B . Then $f(f^{-1}(B)) \supset B$.

Now also note that $A = f^{-1}(B)$. Then $f(A) \supset B$.

Therefore $f(A) = B$. Hence f is surjective.

(c) Is the necessary condition sufficient?

• Suppose that f is surjective.

Let U be a subset of B . [We want to deduce that for any object y , if $y \in U$ then $y \in f(f^{-1}(U))$.]

* Pick any object y . Suppose $y \in U$. [We want to deduce that $y \in f(f^{-1}(U))$.]

Since f is surjective, there exists some $x \in A$ such that $y = f(x)$.

Since $y = f(x)$ and $y \in U$, we have $x \in f^{-1}(U)$.

Since $y = f(x)$ and $x \in f^{-1}(U)$, we have $y \in f(f^{-1}(U))$.

It follows that $f(f^{-1}(U)) \supset U$.

4. Conclusion in this investigation? The statement (★) holds:

(★) Let A, B be sets and $f : A \rightarrow B$ be a function. The following statements are equivalent:

(★₁) f is surjective.

(★₂) For any subset U of B , $f(f^{-1}(U)) \supset U$.

(★₃) For any subset U of B , $f(f^{-1}(U)) = U$.