MATH1050 Image sets, pre-image sets of intervals for continuous real-valued functions of one real-variable.

1. **Definition.**
   Let S be a subset of $\mathbb{R}$. The set S is said to be an **interval in** $\mathbb{R}$ *if any one of the statements below hold:*
   (a) $S = \emptyset$.
   (b) $S = \{x \in \mathbb{R} : a < x\}$ *for some* $a \in \mathbb{R}$.
   (c) $S = \{x \in \mathbb{R} : a \leq x\}$ *for some* $a \in \mathbb{R}$.
   (d) $S = \{x \in \mathbb{R} : x < b\}$ *for some* $b \in \mathbb{R}$.
   (e) $S = \{x \in \mathbb{R} : x \leq b\}$ *for some* $b \in \mathbb{R}$.
   (f) $S = \{x \in \mathbb{R} : a < x < b\}$ *for some* $a, b \in \mathbb{R}$.
   (g) $S = \{x \in \mathbb{R} : a \leq x < b\}$ *for some* $a, b \in \mathbb{R}$.
   (h) $S = \{x \in \mathbb{R} : a < x \leq b\}$ *for some* $a, b \in \mathbb{R}$.
   (i) $S = \{x \in \mathbb{R} : a \leq x \leq b\}$ *for some* $a, b \in \mathbb{R}$.
   (j) $S = \mathbb{R}$.

   Remark on notations and terminologies. We write:
   - $(a, +\infty) = \{x \in \mathbb{R} : a < x\}$
   - $[a, +\infty) = \{x \in \mathbb{R} : a \leq x\}$
   - $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$
   - $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$
   - $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
   - $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
   - $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
   - $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

   Each of the numbers a, b is called an endpoint of the interval concerned. If the interval concerned contains all its endponts as its elements, it is said to be a **closed interval**. Where it contains none, it is said to be an **open interval**. If the interval is bounded in $\mathbb{R}$, it is said to be a **bounded interval**. If it is not bounded in $\mathbb{R}$, it is said to be an **unbounded interval**.

2. **Theorem (1). (Characterization of intervals.)**
   Let S be a subset of $\mathbb{R}$. The statements (I1), (I2) *below are equivalent:*
   (I1) *S is an interval.*
   (I2) *For any $x, y \in S$, for any $u \in \mathbb{R}$, if $x \leq u \leq y$ then $u \in S$.*

3. **Outline of a proof of Theorem (1).**
   The argument for '(I1)$\Longrightarrow$(I2)' is tedious but straightforward. Below is an outline of the argument for '(I2)$\Longrightarrow$(I1)':
   (a) Let S be a subset of $\mathbb{R}$. Suppose S satisfies (I2).
   If $S = \emptyset$ then S is an interval.
   From now on, suppose $S \neq \emptyset$.
   (b) Suppose S is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$. Then, by applying (I2), we deduce $\mathbb{R} \subset S$. It follows that $S = \mathbb{R}$.
   (c) Suppose S is bounded above in $\mathbb{R}$, and S is not bounded below in $\mathbb{R}$.
      i. Take some $c \in S$. By applying (I2), we deduce that $(-\infty, c] \subset S$.
      ii. Since S is non-empty and is bounded above in $\mathbb{R}$, S has a supremum in $\mathbb{R}$, which we denote by b.
      iii. Since b is an upper bound of S in $\mathbb{R}$, we have $S \subset (-\infty, b]$.
      iv. By applying (I2), we deduce that $(c, b) \subset S$. (Why? Pick any $u \in (c, b)$. Since $k < b$, u is not an upper bound of S in $\mathbb{R}$. Then there exists some $v \in S$ such that $v > u$. Since $v \in S$, we have $v \leq b$. Then $c < k < v$. By (I2), $k \in S$.)
      v. It follows that $(-\infty, b) \subset S \subset (-\infty, b]$. Then $S = (-\infty, b)$ or $(-\infty, b]$.
   (d) Suppose S is bounded below in $\mathbb{R}$, and S is not bounded above in $\mathbb{R}$.
   Modifying the argument above, we deduce that S has an infimum in $\mathbb{R}$, which we denote by a, and that furthermore, $(a, +\infty) \subset S \subset [a, +\infty)$. Then $S = (a, +\infty)$ or $[a, +\infty)$.
   (e) Suppose S is bounded below in $\mathbb{R}$ and bounded above in $\mathbb{R}$.
   Modifying the argument above, we deduce that S has an infimum and a supremum in $\mathbb{R}$, we denote by a, b respectively, and that furthermore, $(a, b) \subset S \subset [a, b]$.
   Then $S = (a, b)$ or $S = (a, b]$ or $S = [a, b)$ or $S = [a, b]$.

4. **Theorem (2).**
   Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.
   Suppose g is continuous on J. Then $g(J)$ is an interval.
   Remark. The proof of Theorem (2) relies on Theorem (1) and the Intermediate Value Theorem.

5. **Intermediate Value Theorem.**

   Let $a, b \in \mathbb{R}$, with $a < b$. Let $h : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $h(a) \neq h(b)$. Suppose h is continuous on $[a, b]$. Then, for any $\gamma \in \mathbb{R}$, if γ is strictly between $h(a)$ and $h(b)$ then there exists some $c \in (a, b)$ such that $h(c) = \gamma$.

6. **Proof of Theorem (2).**

   Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

   Suppose g is continuous on J.

   [We want to verify that for any $s, t \in g(J)$, for any $u \in \mathbb{R}$, if $s \leq u \leq t$ then $u \in g(J)$.]

   Pick any $s, t \in g(J)$. Pick any $u \in \mathbb{R}$. Suppose $s \leq u \leq t$.

   If $s = t$ or $u = s$ or $u = t$, then $u \in g(J)$ trivially. From now on suppose $s < u < t$.

   By the definition of $g(J)$, there exists some $a, b \in J$ such that $s = g(a)$ and $t = g(b)$. Since $s < t$, we have $a \neq b$.

   By assumption g is continuous on J. Since J is an interval and $a, b \in J$, the closed and bounded interval I with endpoints a, b lies entirely in J. Then g is continuous on I.

   By the Intermediate Value Theorem, there exists some c strictly between a, b such that $u = g(c)$. Then $u \in g(J)$.

   Now by Theorem (1), $g(J)$ is an interval.

7. **Theorem (3).**

   Let K be a closed and bounded interval in $\mathbb{R}$, and $g : K \longrightarrow \mathbb{R}$ be a function.

   Suppose g is continuous on K. Then $g(K)$ is a closed and bounded interval. Moreover, the endpoints of $g(K)$ are respectively the least element and the greatest element of $g(K)$.

   Remark. The proof of Theorem (3) relies on Theorem (2), the Intermediate Value Theorem and the Existence-of-Extremum Theorem for continuous functions.

   Further remark. The statements below are false:

   (a) *Let J be an open interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.*
   Suppose g is continuous on J. Then $g(J)$ is an open interval.

   (b) *Let J be a closed interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.*
   Suppose g is continuous on J. Then $g(J)$ is a closed interval.

   (c) *Let J be a bounded interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.*
   Suppose g is continuous on J. Then $g(J)$ is a bounded interval.

   (d) *Let J be an unbounded interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.*
   Suppose g is continuous on J. Then $g(J)$ is an unbounded interval.

8. **Definition. (Absolute extrema for real-valued functions of one real variable.)**

   Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely. Let p be a point in I.

   (a) *h is said to* **attain absolute maximum at p on I** *if for any $x \in I$, the inequality $h(x) \leq h(p)$ holds. The number $h(p)$ is called the* **absolute maximum value of h on I.**

   (b) *h is said to* **attain absolute minimum at p on I** *if for any $x \in I$, the inequality $h(x) \geq h(p)$ holds. The number $h(p)$ is called the* **absolute minimum value of h on I.**

   (c) *h is said to* **attain (absolute) extremum at p on I** *if h attains absolute maximum at p or h attains absolute minimum at p.*

   Existence-of-Extremum Theorem for continuous functions.

   Let I be a closed and bounded interval in $\mathbb{R}$, and $f : I \longrightarrow \mathbb{R}$ be a function.

   Suppose f is continuous on I. Then there exist some $p, q \in I$ such that f attains absolute minimum at p on I and f attains absolute maximum at q on I.

   Remark. The key step in the proof of the Existence-of-Extremum Theorem for continuous functions is to prove that the function f is bounded, in the sense that there exist some positive real number C such that for any $x \in I$, $|f(x)| \leq C$. The technical detail for this step is beyond the scope of this course: it relies on ideas that will be introduced in your *analysis* course.

9. **Proof of Theorem (3).**

   Let K be a closed and bounded interval in $\mathbb{R}$, and $g : K \longrightarrow \mathbb{R}$ be a function.

   Suppose g is continuous on K.

   By Theorem (2), $g(K)$ is an interval.

   By the Existence-of-Extremum Theorem for continuous functions, there exist some $p, q \in K$ such that g attains absolute minimum at p on K and g attains absolute maximum at q on K.

   We verify that $g(K) = [g(p), g(q)]$:

   - Pick any $y \in [g(p), g(q)]$. Then $g(p) \leq y \leq g(q)$. If $y = g(p)$ or $y = g(q)$ then $y \in g(K)$.
     From now on suppose $g(p) < y < g(q)$. Then by the Intermediate Value Theorem, there exists some x strictly between p and q such that $y = g(x)$.
     Since K is an interval, $p, q \in K$ and x is strictly between p, q, we have $x \in K$.
     Then $y \in g(K)$.
   - Suppose $v \in g(K)$. Then there exists some $u \in K$ such that $v = g(u)$.
     Since g attains absolute minimum at p on K, we have $v = g(u) \geq g(p)$
     Since g attains absolute maximum at q on K, we have $v = g(u) \leq g(q)$.
     Then $g(p) \leq v \leq g(q)$. Therefore $v \in [g(p), g(q)]$.

   Hence $g(K)$ is the closed and bounded interval $[g(p), g(q)]$, whose least element and greatest element are respectively $g(p), g(q)$.

10. **Definition.**

    Let S be a subset of $\mathbb{R}$.

    (a) *S is said to be* **open in** $\mathbb{R}$ *if for any $x \in S$, there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subset S$.*

    (b) *S is said to be* **closed in** $\mathbb{R}$ *if $\mathbb{R} \backslash S$ is open in $\mathbb{R}$.*

    Remark.

    (a) $\emptyset$ is open in $\mathbb{R}$ and is closed in $\mathbb{R}$.

    (b) $\mathbb{R}$ is open in $\mathbb{R}$ and is closed in $\mathbb{R}$.

    (c) Every open interval in $\mathbb{R}$ is open in $\mathbb{R}$.

    (d) Every closed interval in $\mathbb{R}$ is closed in $\mathbb{R}$.

11. **Definition.**

    Let A be a subset of $\mathbb{R}$, and S be a subset of A.

    (a) *S is said to be* **open in** *A if for any $x \in S$, there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \cap A \subset S$.*

    (b) *S is said to be* **closed in** *A if $A \backslash S$ is open in A.*

    Lemma (4).

    Let A be a subset of $\mathbb{R}$, and S be a subset of A. The statements below are logically equivalent:

    (a) *S is open in A.*

    (b) *There exists some subset T of $\mathbb{R}$ such that T is open in $\mathbb{R}$ and $S = T \cap A$.*

    Remark. The proof of Lemma (4) is easy.

12. **Theorem (5).**

    Let D be a subset of $\mathbb{R}$, and $f : D \longrightarrow \mathbb{R}$ be a function.

    The statements below are logically equivalent:

    (a) *f is continuous on D.*

    (b) *For any subset U of $\mathbb{R}$, if U is open in $\mathbb{R}$ then $f^{-1}(U)$ is open in D.*

    (c) *For any subset J of $\mathbb{R}$, if J is an open interval in $\mathbb{R}$ then $f^{-1}(J)$ is open in D.*

Remark. Theorem (5) is a straightforward consequence of the definition of pre-image set, the definition of open set in $\mathbb{R}$, and the (formal) definition for the notion of continuity.

Definition.

Let A be a subset of $\mathbb{R}$, and $h : A \longrightarrow \mathbb{R}$ be a function. Let $c \in A$.

h is said to be **continuous at** *c if the statement* (CT) *holds:*

(CT) *For any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $x \in A$, if $|x - c| < \delta$ then $|h(x) - h(c)| < \varepsilon$.*

Furthermore, h is said to be **continuous on** *D if h is continuous at every point of D.*

13. **Proof of Theorem (5).**

Let D be a subset of $\mathbb{R}$, and $f : D \longrightarrow \mathbb{R}$ be a function.

- [(a)$\Longrightarrow$(b)?]

  Suppose f is continuous on D.

  [We want to prove that for any subset U of $\mathbb{R}$, if U is open in $\mathbb{R}$ then $f^{-1}(U)$ is open in D.]

  Let U be a subset of $\mathbb{R}$. Suppose U is open in $\mathbb{R}$.

  [We want to verify that $f^{-1}(U)$ is open in $\mathbb{R}$.]

  Pick any $c \in f^{-1}(U)$. By the definition of pre-image set, we have $f(c) \in U$.
  Since U is open in $\mathbb{R}$, there exists some $\eta > 0$ such that $(f(c) - \eta, f(c) + \eta) \subset U$.
  By continuity, for the same $\eta > 0$, there exists some $\delta > 0$ such that for any $x \in D$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \eta$.
  We verify that $(c - \delta, c + \delta) \cap D \subset f^{-1}(U)$:

  Pick any $x \in (c - \delta, c + \delta) \cap D$. We have $x \in D$ and $|x - c| < \delta$. Then (by continuity,) $|f(x) - f(c)| < \eta$. Therefore $f(x) \in (f(c) - \eta, f(c) + \eta)$. Hence $f(x) \in U$.
  By the definition of pre-image set, we have $x \in f^{-1}(U)$.

  It follows that $f^{-1}(U)$ is open in $\mathbb{R}$.

- [(b)$\Longrightarrow$(c)?]

  Suppose that for any subset U of $\mathbb{R}$, if U is open in $\mathbb{R}$ then $f^{-1}(U)$ is open in D.

  [We want to prove that for any subset J of $\mathbb{R}$, if J is an open interval in $\mathbb{R}$ then $f^{-1}(J)$ is open in D.]

  Let J be a subset of $\mathbb{R}$. Suppose J is an open interval in $\mathbb{R}$.
  Note that J is open in $\mathbb{R}$. Then, by assumption, $f^{-1}(J)$ is open in D.

- [(c)$\Longrightarrow$(a)?]

  Suppose that for any subset J of $\mathbb{R}$, if J is an open interval in $\mathbb{R}$ then $f^{-1}(J)$ is open in D.

  [We want to prove that f is continuous on D. This amounts to verify that for any $c \in D$, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $x \in D$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.]

  Pick any $c \in D$. Pick any $\varepsilon > 0$. Write $J = (f(c) - \varepsilon, f(c) + \varepsilon)$. Note that J is an open interval in $\mathbb{R}$.
  By assumption, $f^{-1}(J)$ is open in D.
  Since $f(c) \in J$, we have $c \in J$ by the definition of pre-image set.
  Then, by the definition of open set, there exists some $\delta > 0$ such that $(c - \delta, c + \delta) \cap D \in f^{-1}(J)$.

  [We now ask: Is it true that for any $x \in D$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$?]

  Pick any $x \in D$. Suppose $|x - c| < \delta$. Then $x \in (c - \delta, c + \delta)$ and $x \in D$. Therefore $x \in (c - \delta, c + \delta) \cap D$. Hence $x \in f^{-1}(J)$.
  Now by the definition of pre-image set, we have $f(x) \in J$. Then $|f(x) - f(c)| < \varepsilon$.
  It follows that f is continuous at c.