MATH1050 Image sets and pre-image sets

1. **Definition**.
   Let A, B be sets and $f : A \longrightarrow B$ be a function from A to B. Let S be a subset of A. The **image set of the set S under the function f** *is defined to be the set*

$$\{y \in B : \text{There exists some } x \in S \text{ such that } y = f(x)\}$$

   (or $\{y \in B : y = f(x) \text{ for some } x \in S\}$, or $\{f(x) \mid x \in S\}$). It is denoted by $f(S)$.

   Remarks.

   (1) **Terminology.** We write $f(A) = \{y \in B : y = f(x) \text{ for some } x \in A\}$.
   The set $f(A)$ is called the **image of the function f**.

   (2) **Warning.** '$f(S)$' should be understood as one symbol which is the name of some subset of B whose 'content' depends on f and S, according to the predicate

$$\text{'there exists some } x \in S \text{ such that } y = f(x)\text{'}$$

   for its construction via the Method of Specification.

2. **Example of pictorial visualizations of image sets**.
   Let $A = \{k, \ell, m, n, p, q, r, \cdots\}$, $B = \{c, d, e, g, h, i, j, \cdots\}$, and $f : A \longrightarrow B$ be the function defined by $f(k) = d$, $f(\ell) = f(m) = f(n) = e$, $f(p) = g$, $f(q) = f(r) = h$, ...

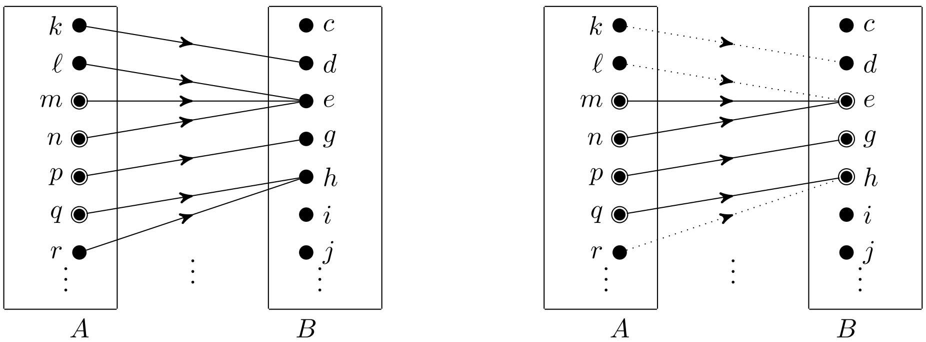
   Let $S = \{m, n, p, q\}$.

   We have $f(S) = \{f(m), f(n), f(p), f(q)\} = \{e, g, h\}$.

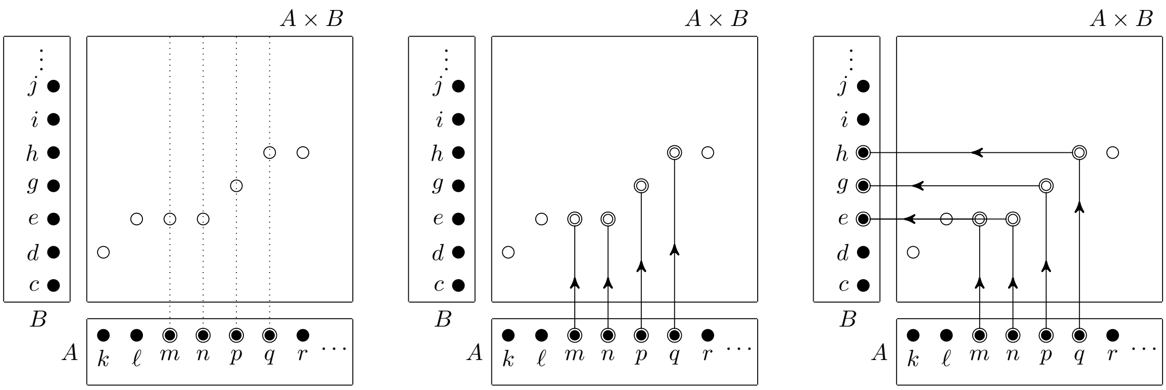
   (a) 'Blobs-and-arrows diagram'.
   Two equivalent ways to see what $f(S)$ is:

   (i) Examine each $y \in B$. Ask if it is targetted by some $x \in S$ via f. If yes, collect it; if no, throw away.

   (ii) Simply collect the target of every $x \in S$.



   (b) 'Coordinate plane diagram'.



3. **Definition of pre-image sets.**

Let A, B be sets and $f : A \longrightarrow B$ be a function from A to B. Let U be a subset of B. The **pre-image set of the set U under the function f** *is defined to be the set*

$$\{x \in A : \text{There exists some } y \in U \text{ such that } y = f(x)\}$$

(or $\{x \in A : y = f(x) \text{ for some } y \in U\}$, or $\{x \in A : f(x) \in U\}$). It is denoted by $f^{-1}(U)$.

Warnings.

(1) '$f^{-1}(U)$' should be understood as one symbol which is the name of some subset of A whose 'content' depends on f and U, according to the predicate

$$\text{'there exists some } y \in U \text{ such that } y = f(x)\text{'}$$

for its construction via the Method of Specification.

(2) The presence of the chain of symbols 'f^{-1}' in '$f^{-1}(U)$' does not give any hint as to whether f is 'invertible'/bijective as a function or not.

4. **Example of pictorial visualizations of pre-image sets.**

Let $A = \{k, \ell, m, n, p, q, r, \cdots\}$, $B = \{c, d, e, g, h, i, j, \cdots\}$, and $f : A \longrightarrow B$ be the function defined by $f(k) = d$, $f(\ell) = f(m) = f(n) = e$, $f(p) = g$, $f(q) = f(r) = h$, ...

Let $U = \{e, g, i\}$.

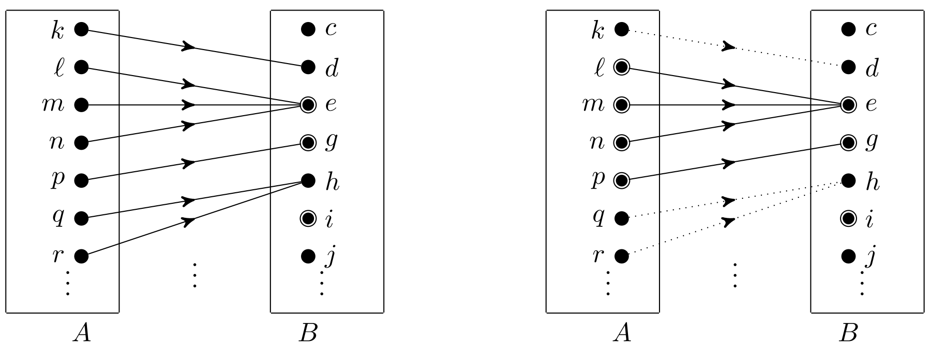
We have $f^{-1}(U) = \{\ell, m, n, p\} = \{\ell, m, n\} \cup \{p\} \cup \emptyset = f^{-1}(\{e\}) \cup f^{-1}(\{g\}) \cup f^{-1}(\{i\})$.

(a) 'Blobs-and-arrows diagram'.

Two equivalent ways to see what $f^{-1}(U)$ is:

(i) Examine each $x \in A$. Ask if it targets some $y \in U$ via f. If yes, collect it; if no, throw away.

(ii) For each $y \in U$, collect whatever in A that targets it, and then 'take union'.



(b) 'Coordinate plane diagram'.

For each $y_0 \in B$, to determine the pre-image set $f^{-1}(\{y_0\})$ of the singleton $\{y_0\}$ under f is the same as to determine the **solution set** of the equation $y_0 = f(x)$ with unknown x in A. We call this set the **level set** at y_0 of the function f. So $f^{-1}(U)$ is the 'union' of the level sets of the y's in U of the function f.