MATH1050 Surjectivity and injectivity for 'simple' complex-valued functions of one complex variable

1. **Example (1).**

   Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2$ for any $z \in \mathbb{C}$.

   Is f surjective? Yes. Justification:

   * $*$ [What to verify? For any $\zeta \in \mathbb{C}$, there exists some $z \in \mathbb{C}$ such that $f(z) = \zeta$.]

     Pick any $\zeta \in \mathbb{C}$. Note that $\zeta = 0$ or $\zeta \neq 0$.

     (†) Suppose $\zeta = 0$. We have $0 \in \mathbb{C}$ and $f(0) = 0 = \zeta$.

     ‡ Suppose $\zeta \neq 0$. [Try to name some appropriate $z \in \mathbb{C}$ satisfying $f(z) = \zeta$. Roughwork?]

     There exists some $\theta \in \mathbb{R}$ such that $\zeta = |\zeta|(\cos(\theta) + i\sin(\theta))$.

     Take $z = \sqrt{|\zeta|} \cdot \left(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\right)$. By definition, $z \in \mathbb{C}$.

     $$\begin{aligned} f(z) = z^2 &= \left[\sqrt{|\zeta|} \cdot \left(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\right)\right]^2 \\ &= (\sqrt{|\zeta|})^2 \cdot \left(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\right)^2 = |\zeta|(\cos(\theta) + i\sin(\theta)) = \zeta \end{aligned}$$

     It follows that f is surjective.

   Remark. Contrast the above result with this statement: *The function $p : \mathbb{R} \longrightarrow \mathbb{R}$ given by $p(x) = x^2$ for any $x \in \mathbb{R}$ is not surjective.*

2. **Example (2).**

   Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $g(z) = z^3$ for any $z \in \mathbb{C}$.

   Is g injective? No. Justification:

   * $*$ [What to verify? There exists some $z, w \in \mathbb{C}$ such that $z \neq w$ and $g(z) = g(w)$.]

     [Try to name some appropriate distinct $z, w \in \mathbb{C}$ satisfying $g(z) = g(w)$. Roughwork?]

     Take $z = 1$, $w = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$. ($w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.)

     Note that $z, w \in \mathbb{C}$ and $z \neq w$.

     $g(z) = 1^3 = 1$.

     $g(w) = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)^3 = \cos(2\pi) + i\sin(2\pi) = 1$.

     Then $g(z) = g(w)$.

     It follows that g is not injective.

   Remark. Contrast the above result with this statement: *The function $q : \mathbb{R} \longrightarrow \mathbb{R}$ given by $q(x) = x^3$ for any $x \in \mathbb{R}$ is injective.*

3. **Example (3).**

   Let $n \in \mathbb{N}\backslash\{0, 1\}$, and $h : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $h(z) = z^n$ for any $z \in \mathbb{C}$.

   Is h surjective? Is h injective?

   The respective answers and justifications are similar to what we have done above.

4. **Example (4).**

   Let $a, b \in \mathbb{C}$. Suppose $a \neq 0$. Define the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ by $f(z) = az + b$ for any $z \in \mathbb{C}$.

   Is f surjective? Yes. Justification:

   * $*$ [What to verify? For any $\zeta \in \mathbb{C}$, there exists some $z \in \mathbb{C}$ such that $f(z) = \zeta$.]

     Pick any $\zeta \in \mathbb{C}$.

     [Try to name some appropriate $z \in \mathbb{C}$ satisfying $f(z) = \zeta$. Roughwork?]

     Take $z = \frac{\zeta - b}{a}$. By definition $z \in \mathbb{C}$. $f(\zeta) = a \cdot \frac{\zeta - b}{a} + b = \zeta$.

     It follows that f is surjective.

   Is f injective? Yes. Justification:

* [What to verify? For any $z, w \in \mathbb{C}$, if $f(z) = f(w)$ then $z = w$.]
  Pick any $z, w \in \mathbb{C}$. Suppose $f(z) = f(w)$. [Try to deduce $z = w$.]
  Then $az + b = aw + b$. Therefore $az = aw$. Since $a \neq 0$, $z = w$.
  It follows that f is injective.

5. **Example (5).**
   Let $a, b, c \in \mathbb{C}$. Suppose $a \neq 0$. Define the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ by $f(z) = az^2 + bz + c$ for any $z \in \mathbb{C}$.

   Write $\gamma = -\dfrac{b}{2a}$, $\Delta = b^2 - 4ac$. Note that $f(z) = a(z-\gamma)^2 - \dfrac{\Delta}{4a}$ for any $z \in \mathbb{C}$.

   Is f surjective? Yes. Justification:

   * [What to verify? For any $\zeta \in \mathbb{C}$, there exists some $z \in \mathbb{C}$ such that $f(z) = \zeta$.]
     Pick any $\zeta \in \mathbb{C}$.
     [Try to name some appropriate $z \in \mathbb{C}$ satisfying $f(z) = \zeta$. Roughwork?]
     Note that $\zeta = -\dfrac{\Delta}{4a}$ or $\zeta \neq -\dfrac{\Delta}{4a}$.

     (†) Suppose $\zeta = -\dfrac{\Delta}{4a}$.
     Take $z = \gamma$.
     $\gamma \in \mathbb{C}$, and $f(z) = f(\gamma) = a \cdot 0 - \dfrac{\Delta}{4a} = \zeta$.

     (‡) Suppose $\zeta \neq -\dfrac{\Delta}{4a}$. Define $\alpha = \dfrac{1}{a}\left(\zeta + \dfrac{\Delta}{4a}\right)$. By definition, $\alpha \in \mathbb{C}\backslash\{0\}$.
     There exists some $\theta \in \mathbb{R}$ such that $\alpha = |\alpha|(\cos(\theta) + i\sin(\theta))$.
     Take $z = \gamma + \sqrt{|\alpha|} \cdot \left(\cos(\dfrac{\theta}{2}) + i\sin(\dfrac{\theta}{2})\right)$. By definition $z \in \mathbb{C}$.

     $$\begin{aligned} f(z) = a(z-\gamma)^2 - \frac{\Delta}{4a} &= a\left[\sqrt{|\alpha|} \cdot \left(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\right)\right]^2 - \frac{\Delta}{4a} \\ &= a|\alpha|(\cos(\theta) + i\sin(\theta)) - \frac{\Delta}{4a} = a\alpha - \frac{\Delta}{4a} = \zeta \end{aligned}$$

     It follows that f is surjective.

   Is f injective? No. Justification:

   * [What to verify? There exist some $z, w \in \mathbb{C}$ such that $z \neq w$ and $f(z) = f(w)$.]
     [Try to name some appropriate distinct $z, w \in \mathbb{C}$ satisfying $f(z) = f(w)$. Roughwork?]
     Take $z = \gamma + 1$, $w = \gamma - 1$. Note that $z, w \in \mathbb{C}$ and $z \neq w$. $f(z) = a - \dfrac{\Delta}{4a} = f(w)$.
     It follows that f is not injective.

6. **Polynomial functions on $\mathbb{C}$.**
   We introduce these definitions:

   (a) *Let $n \in \mathbb{N}$. A* **degree-n polynomial with complex coefficients and with indeterminate** *z is an expression of the form $a_n z^n + \cdots + a_1 z + a_0$ in which $a_0, a_1, \cdots, a_n \in \mathbb{C}$ and $a_n \neq 0$.*

   (b) *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function. f is said to be a* **degree-n polynomial function (with complex coefficients) on $\mathbb{C}$** *if there exist some $a_0, a_1, \cdots, a_n \in \mathbb{C}$ such that $a_n \neq 0$ and $f(z) = a_n z^n + \cdots + a_1 z + a_0$ for any $z \in \mathbb{C}$.*

   The examples above are special cases of these results:

   Theorem (1).
   Let $n \in \mathbb{N}\backslash\{0, 1\}$. Every degree-$n$ polynomial function on $\mathbb{C}$ is surjective.

   Theorem (2).
   Let $n \in \mathbb{N}\backslash\{0, 1\}$. Every degree-$n$ polynomial function on $\mathbb{C}$ is not injective.

   Theorem (1) is logically equivalent to the **Fundamental Theorem of Algebra**:
   Every non-constant polynomial with complex coefficient has a root in $\mathbb{C}$.

   Assuming the validity of Theorem (1), we can deduce Theorem (2) easily, with the help of the **Factor Theorem** (whose 'real version' you have already learnt at school and may be carried in verbatim to the 'complex situation' here):

   Let $\alpha \in \mathbb{C}$, and $p(z)$ be a degree-n polynomial (with complex coefficients). Suppose α is a root of $p(z)$. Then there is a degree-$(n-1)$ polynomial $q(z)$ (with complex coefficients) so that $p(z) = (z-\alpha)q(z)$ as polynomials.