1. **Families.**

The notion of infinite sequences of real numbers can be generalized to the notion of families for collections of objects.

Definition.

Suppose I, B *are sets, and* $\varphi : I \longrightarrow B$ *is a function.*

Then we say φ *is a* **family in** B**, indexed by** I.

The set I is referred to as the **index set** for this family, and the set $\{x \in B : x = \varphi(t) \text{ for some } t \in I\}$ is called the **set of all terms** for this family.

Remarks.

(a) This definition generalizes the notion of infinite sequences of real numbers. When $I = \mathbb{N}$, a family in B is just an infinite sequence in B; when furthermore $B = \mathbb{R}$, it is just an infinite sequence of real numbers.

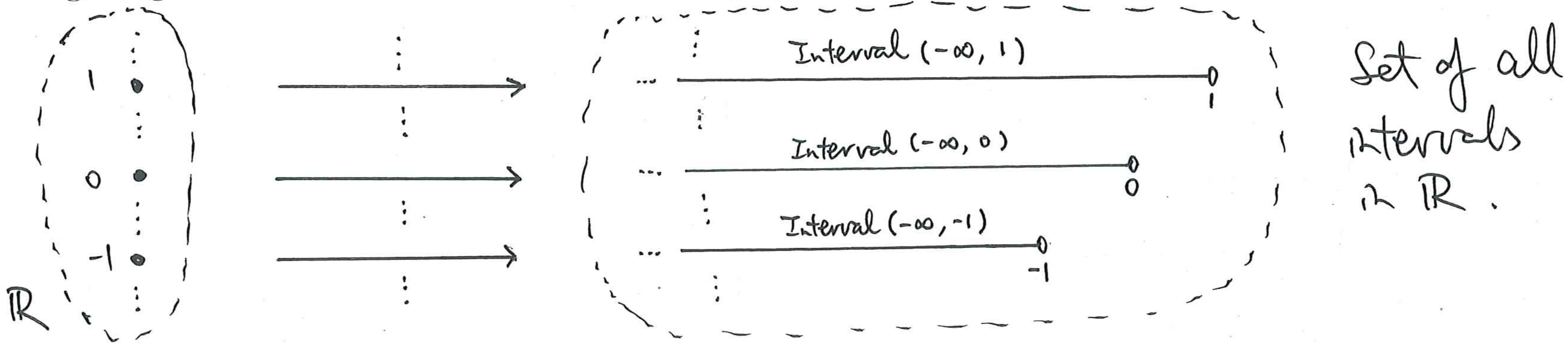
(b) We imitate the notations for infinite sequences when we regard the function φ from I to B as a family in B with index set I. We suppress the symbols φ, B, and present this family as, say, $\{x_\alpha\}_{\alpha \in I}$, in which the symbol x_α stands for $\varphi(\alpha)$ for each $\alpha \in I$.

(c) What is the point of this definition?

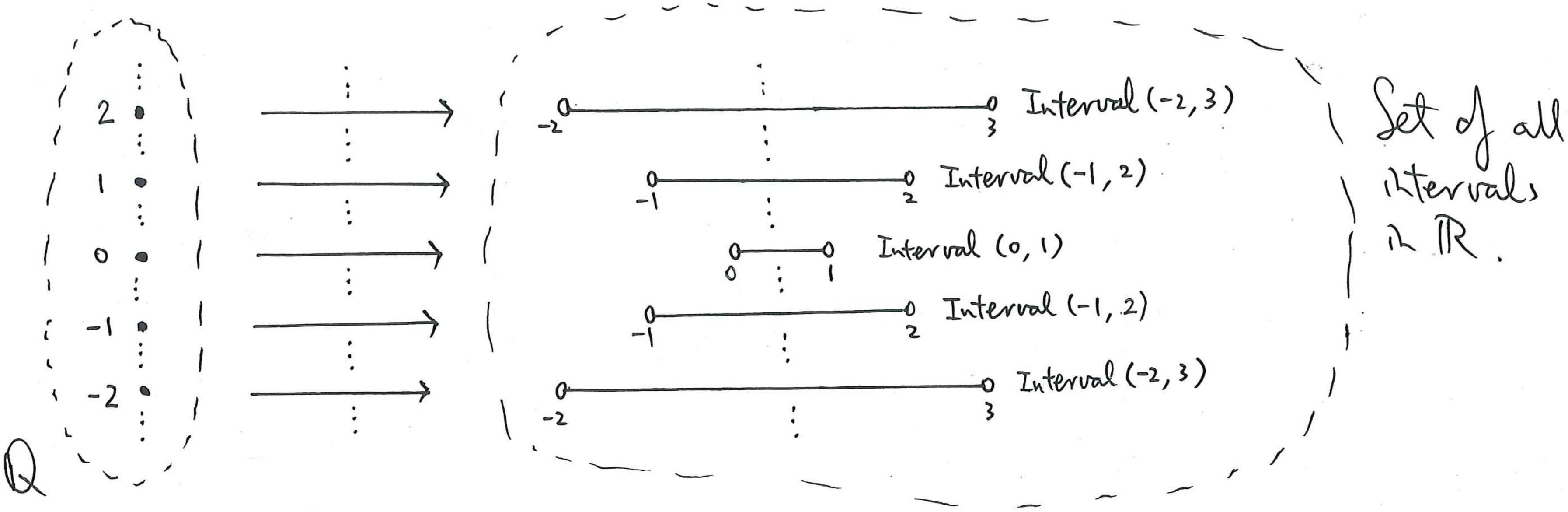
It gives us the flexibility to regard the same family in B, say, $\{x_\alpha\}_{\alpha \in I}$, as a family in C which contains B as a subset, whenever it is convenient for us to do so.

Examples.

(a) $\{(-\infty, u)\}_{u\in\mathbb{R}}$ stands for the function with domain $\mathbb{R}$ and range, say, the set of all intervals, assigning each real number u to the open interval $(-\infty, u)$.



(b) $\{(-|a|, |a|+1)\}_{a\in\mathbb{Q}}$ stands for the function with domain $\mathbb{Q}$ and range, say, the set of all intervals, assigning each rational number a to the open interval $(-|a|, |a|+1)$.



(c) Let p, q be real numbers.

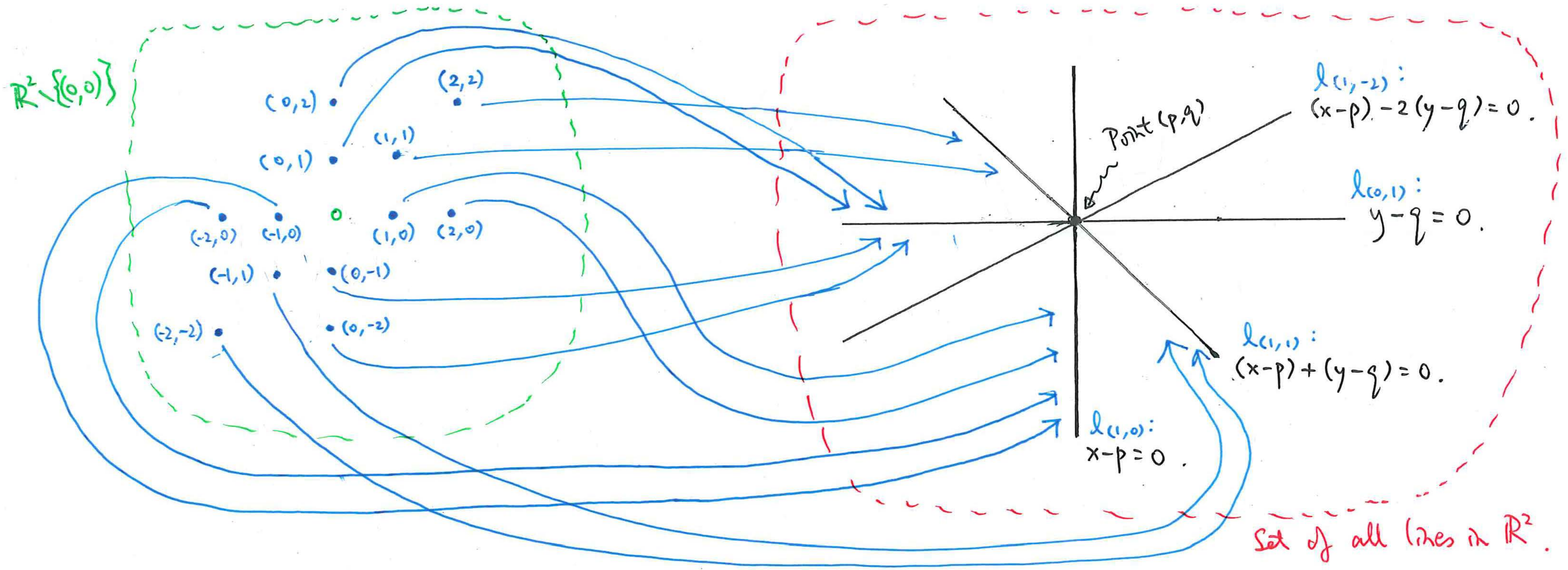
For any real numbers a, b, not both zero, define $\ell_{(a,b)}$ by $\ell_{(a,b)} = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } a(x - p) + b(y - q) = 0\}$.

$\ell_{(a,b)}$ is the line in the infinite plane whose equation is given by $ax + by = ap + bq$.

$\{\ell_{(a,b)}\}_{(a,b)\in\mathbb{R}^2\setminus\{(0,0)\}}$ stands for the function with domain $\mathbb{R}^2\setminus\{(0,0)\}$ and range, say, the set of all lines in the infinite plane, assigning each $(a, b) \in \mathbb{R}^2\setminus\{(0,0)\}$ to the line $\ell_{(a,b)}$.

$\{\ell_{(a,b)}\}_{(a,b)\in\mathbb{R}^2\setminus\{(0,0)\}}$ stands for the function with domain $\mathbb{R}^2\setminus\{(0,0)\}$ and range, say, the set of all lines in the infinite plane, assigning each $(a, b) \in \mathbb{R}^2\setminus\{(0,0)\}$ to the line $\ell_{(a,b)}$.

(The set of all terms of this family is the pencil of all lines passing through (p, q).)



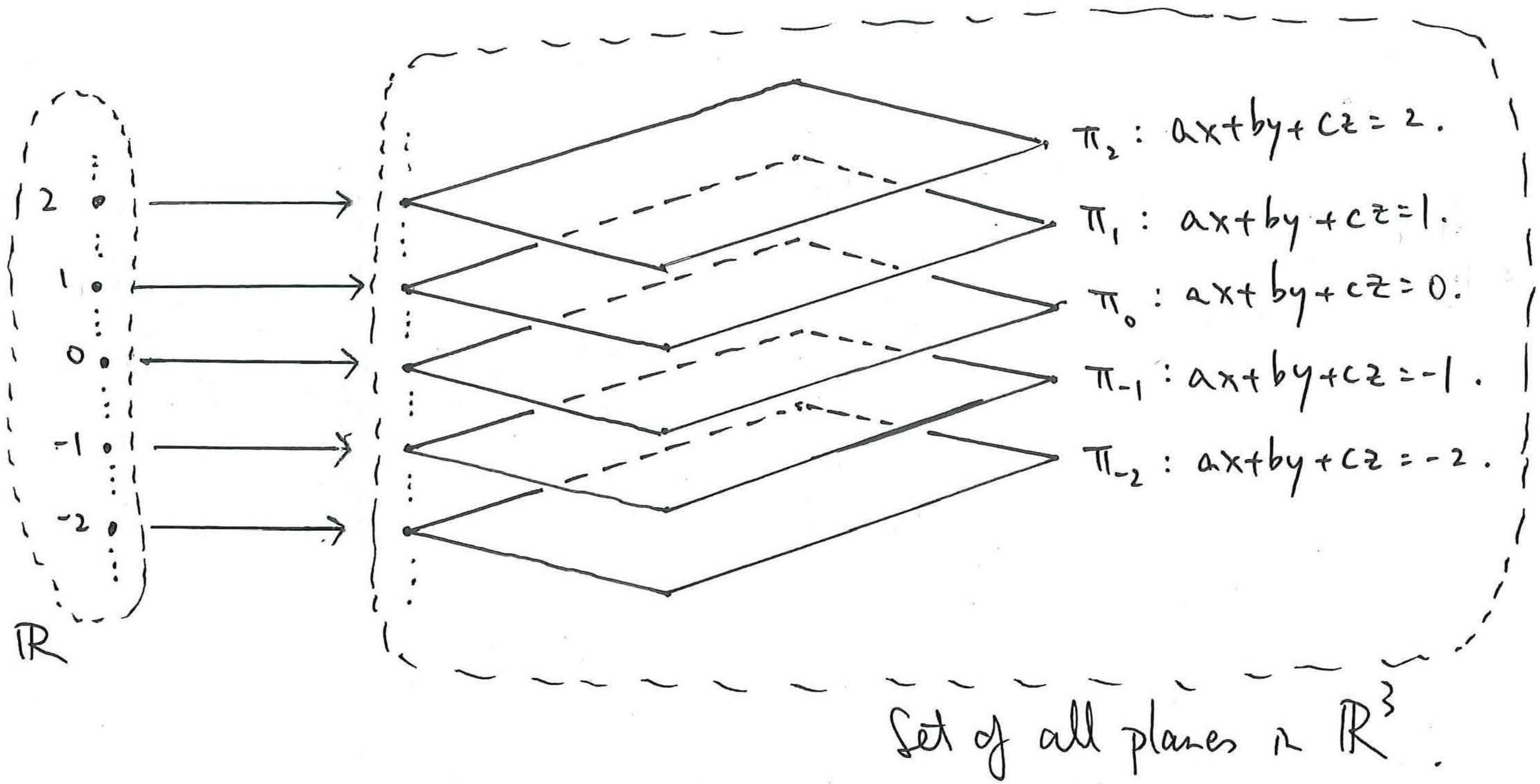
(d) Let a, b, c be real numbers, not all zero.

For any real number d, define π_d by $\pi_d = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } ax + by + cz = d\}$.

π_d is the plane in the infinite space whose equation is given by $ax + by + cz = d$.

$\{\pi_d\}_{d \in \mathbb{R}}$ stands for the function with domain $\mathbb{R}$ and range, say, the set of all planes in the infinite space, assigning each $d \in \mathbb{R}$ to the plane π_d.

(The set of all terms of this family is the pencil of planes all parallel to the plane $ax + by + cz = 0$.)



(e) For each complex number p, for each positive real number r, define

$$C(\zeta, r) = \{z \in \mathbb{C} : |z - p| = r\}.$$

($C(\zeta, r)$ is the circle in the Argand plane with centre p and radius r.)

Let $p \in \mathbb{C}$.

$\{C(p, r)\}_{r \in (0,+\infty)}$ stands for the function with domain $(0, +\infty)$ and range, say, the set of all circles in the Argand plane, assigning each positive real number to the circle $C(p, r)$. (The set of all terms of this family is the pencil of circles concentric at p.)

(f) For each complex number p, for each positive real number r, define

$$D(\zeta, r) = \{z \in \mathbb{C} : |z - p| < r\}.$$

($D(\zeta, r)$ is the 'open disc' in the Argand plane with centre p and radius r.)

$\{D(r, r)\}_{r \in (0,+\infty)}$ stands for the function with domain $(0, +\infty)$ and range, say, the set of all open discs in the Argand plane, assigning each positive real number to the open disc $D(r, r)$.

(g) For each real number α, define the polynomial function $f_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_\alpha(x) = x^2 + \alpha x + 1$ for any $x \in \mathbb{R}$.

$\{f_\alpha\}_{\alpha \in \mathbb{R}}$ stands for the function with domain $\mathbb{R}$ and range, say, the set of all polynomial functions, assigning each real number α to f_α.

2. **Set operations for families of sets.**

The notion of intersection and union for infinite sequences of sets, introduced in the Handout *Universal quantifier and existential quantifier*, can be immediately generalized to families of sets.

Definition.

Let M, I be sets, and $\{S_\alpha\}_{\alpha \in I}$ be a family of subsets of the set M, indexed by I. (For any $\alpha \in I$, S_α is a subset of M.)

(1) *The* **intersection of the family of subsets $\{S_\alpha\}_{\alpha \in I}$ of the set** *M is defined to be the set $\{x \in M : x \in S_\alpha$ for any $\alpha \in I\}$. It is denoted by $\bigcap\limits_{\alpha \in I} S_\alpha$.*

(2) *The* **union of the family of subsets $\{S_\alpha\}_{\alpha \in I}$ of the set** *M is defined to be the set $\{x \in M : x \in S_\alpha$ for some $\alpha \in I\}$. It is denoted by $\bigcup\limits_{\alpha \in I} S_\alpha$.*

Remark. Suppose C is a subset of $\mathfrak{P}(M)$, and we 'index' C by its elements to obtain the family $\{S\}_{S \in C}$. Then the intersection of the set C of subsets of the set M is the intersection of the family $\{S\}_{S \in C}$, and the union of the set C of subsets of the set M is the union of the family $\{S\}_{S \in C}$.

3. Recall this result in the Handout *Universal quantifier and existential quantifier*:

Theorem ($\star$).

Let M be a set and $\{A_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of M.

(1) *Let S be a subset of M. Suppose $S \subset A_n$ for any $n \in \mathbb{N}$. Then $S \subset \bigcap\limits_{n=0}^{\infty} A_n$.*

(2) *Let S be a subset of M. Suppose $S \subset A_n$ for some $n \in \mathbb{N}$. Then $S \subset \bigcup\limits_{n=0}^{\infty} A_n$.*

(3) *Let T be a subset of M. Suppose $A_n \subset T$ for any $n \in \mathbb{N}$. Then $\bigcup\limits_{n=0}^{\infty} A_n \subset T$.*

(4) *Let T be a subset of M. Suppose $A_n \subset T$ for some $n \in \mathbb{N}$. Then $\bigcap\limits_{n=0}^{\infty} A_n \subset T$.*

(5) *Let C be a subset of M. ($\{A_n \cup C\}_{n=0}^{\infty}, \{A_n \cap C\}_{n=0}^{\infty}, \{A_n \backslash C\}_{n=0}^{\infty}, \{C \backslash A_n\}_{n=0}^{\infty}$ are infinite sequences of subsets of M.) The equalities below hold:*

(5a) $\left(\bigcap\limits_{n=0}^{\infty} A_n\right) \cap C = \bigcap\limits_{n=0}^{\infty} (A_n \cap C)$.

(5b) $\left(\bigcup\limits_{n=0}^{\infty} A_n\right) \cup C = \bigcup\limits_{n=0}^{\infty} (A_n \cup C)$.

(5c) $\left(\bigcap\limits_{n=0}^{\infty} A_n\right) \cup C = \bigcap\limits_{n=0}^{\infty} (A_n \cup C)$.

(5d) $\left(\bigcup\limits_{n=0}^{\infty} A_n\right) \cap C = \bigcup\limits_{n=0}^{\infty} (A_n \cap C)$.

(5e) $\left(\bigcap\limits_{n=0}^{\infty} A_n\right) \backslash C = \bigcap\limits_{n=0}^{\infty} (A_n \backslash C)$.

(5f) $\left(\bigcup\limits_{n=0}^{\infty} A_n\right) \backslash C = \bigcup\limits_{n=0}^{\infty} (A_n \backslash C)$.

(5g) $C \backslash \left(\bigcap\limits_{n=0}^{\infty} A_n\right) = \bigcup\limits_{n=0}^{\infty} (C \backslash A_n)$.

(5h) $C \backslash \left(\bigcup\limits_{n=0}^{\infty} A_n\right) = \bigcap\limits_{n=0}^{\infty} (C \backslash A_n)$.

Theorem $(\star)$ can be generalized immediately to Theorem $(\star')$. The proofs of the respective statements are similar.

Theorem $(\star')$.

Let M, I be sets and $\{A_\alpha\}_{\alpha\in I}$ be a family of subsets of M, indexed by I.

(0) *Suppose $I = \emptyset$. Then $\bigcap\limits_{\alpha\in I} A_\alpha = M$ and $\bigcup\limits_{\alpha\in I} A_\alpha = \emptyset$.*

(1) *Let S be a subset of M. Suppose $S \subset A_\alpha$ for any $\alpha \in I$. Then $S \subset \bigcap\limits_{\alpha\in I} A_\alpha$.*

(2) *Let S be a subset of M. Suppose $S \subset A_\alpha$ for some $\alpha \in I$. Then $S \subset \bigcup\limits_{\alpha\in I} A_\alpha$.*

(3) *Let T be a subset of M. Suppose $A_\alpha \subset T$ for any $\alpha \in I$. Then $\bigcup\limits_{\alpha\in I} A_\alpha \subset T$.*

(4) *Let T be a subset of M. Suppose $A_\alpha \subset T$ for some $\alpha \in I$. Then $\bigcap\limits_{\alpha\in I} A_\alpha \subset T$.*

(5) ...

Theorem ($\star$) can be generalized immediately to Theorem ($\star'$). The proofs of the respective statements are similar.

Theorem ($\star'$).

Let M, I be sets and $\{A_\alpha\}_{\alpha\in I}$ be a family of subsets of M, indexed by I. ...

(5) *Let C be a subset of M. ($\{A_\alpha \cup C\}_{\alpha\in I}, \{A_\alpha \cap C\}_{\alpha\in I}, \{A_\alpha\backslash C\}_{\alpha\in I}, \{C\backslash A_\alpha\}_{\alpha\in I}$ are families of subsets of M.) The equalities below hold:*

(5a) $$\left(\bigcap_{\alpha\in I} A_\alpha\right) \cap C = \bigcap_{\alpha\in I}(A_\alpha \cap C).$$

(5b) $$\left(\bigcup_{\alpha\in I} A_\alpha\right) \cup C = \bigcup_{\alpha\in I}(A_\alpha \cup C).$$

(5c) $$\left(\bigcap_{\alpha\in I} A_\alpha\right) \cup C = \bigcap_{\alpha\in I}(A_\alpha \cup C).$$

(5d) $$\left(\bigcup_{\alpha\in I} A_\alpha\right) \cap C = \bigcup_{\alpha\in I}(A_\alpha \cap C).$$

(5e) $$\left(\bigcap_{\alpha\in I} A_\alpha\right) \backslash C = \bigcap_{\alpha\in I}(A_\alpha \backslash C).$$

(5f) $$\left(\bigcup_{\alpha\in I} A_\alpha\right) \backslash C = \bigcup_{\alpha\in I}(A_\alpha \backslash C).$$

(5g) $$C\backslash \left(\bigcap_{\alpha\in I} A_\alpha\right) = \bigcup_{\alpha\in I}(C\backslash A_\alpha).$$

(5h) $$C\backslash \left(\bigcup_{\alpha\in I} A_\alpha\right) = \bigcap_{\alpha\in I}(C\backslash A_\alpha).$$