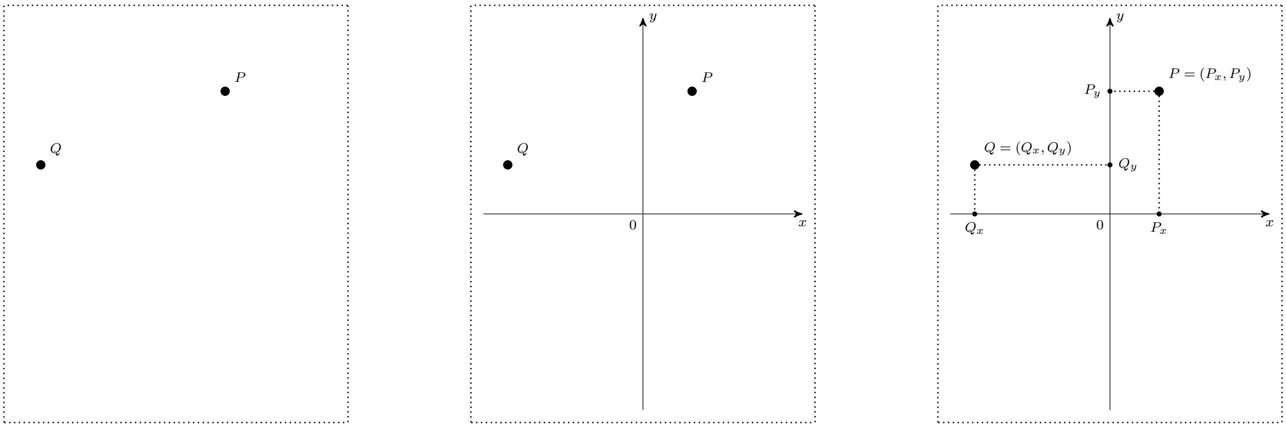
MATH1050 Ordered pairs, ordered triples and cartesian products

1. **Coordinate pairs and Cartesian plane in school mathematics**.

   In school mathematics, we take the notions of coordinate pairs and the Cartesian (coordinate) plane for granted:

   - We fix a pair of mutually perpendicular straight lines in the 'Euclidean plane', each regarded as a copy of the real line $\mathbb{R}$. We call one of them the 'x-axis' and the other the 'y-axis', and call the intersection of the axes the origin.
   - Then we represent each point, say, P, on the plane by a pair of *uniquely determined* real numbers, called the 'x-coordinate' and 'y-coordinate' of the point P. The x-coordinate P_x of P is *uniquely determined* as the number on the x-axis which is the intersection of the x-axis with the line passing through P and being perpendicular to the x-axis. The y-coordinate P_y of P is *uniquely determined* as the number on the y-axis which is the intersection of the n-axis with the line passing through P and being perpendicular to the x-axis. We write $P = (P_x, P_y)$.
   - Now the plane may be regarded to be the same as the set $\{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.



   This is then generalized to coordinate triples and the Cartesian (coordinate) space, and beyond.

   Here we generalize the idea above in the context of set language.

2. **Ordered-ness in set language.**

   Question. *What is the essence in the notion of coordinate pairs in the plane?*

   The essence is contained in this statement below (which we have tacitly assumed since school mathematics):

   - *For any $s, t, u, v \in \mathbb{R}$, ($(s, t) = (u, v)$ iff ($s = u$ and $t = v$)).*

   The bi-conditional '$(s, t) = (u, v)$ *iff* ($s = u$ *and* $t = v$)' conveys the sense of ordered-ness. We are going to borrow it to set language.

   For the moment, we assume it makes sense to talk about the ordered pair (s, t) of any two given objects s, t and refer to s, t as the first, second coordinates of the ordered pair (s, t). For the words 'ordered', 'first', 'second' to make any sense, '$(\cdot, \cdot)$' has to obey Convention ($\sharp$) below:

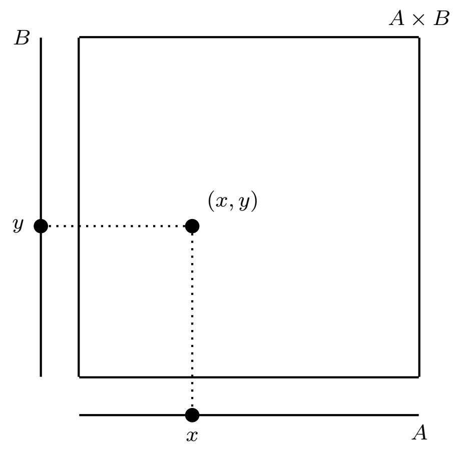
   ($\sharp$) *For any objects x, y, u, v, ($(x, y) = (u, v)$ then ($x = u$ and $y = v$)).*

3. **Cartesian product of two sets.**

With this sense of ordered-ness in mind, it makes sense to define the notion of Cartesian product of two sets:

Definition.

Let A, B be sets. The **cartesian product** $A \times B$ *of the sets A, B is defined to be the set* $\{t \mid$ *There exist some* $x \in A, y \in B$ *such that* $t = (x, y)\}$.



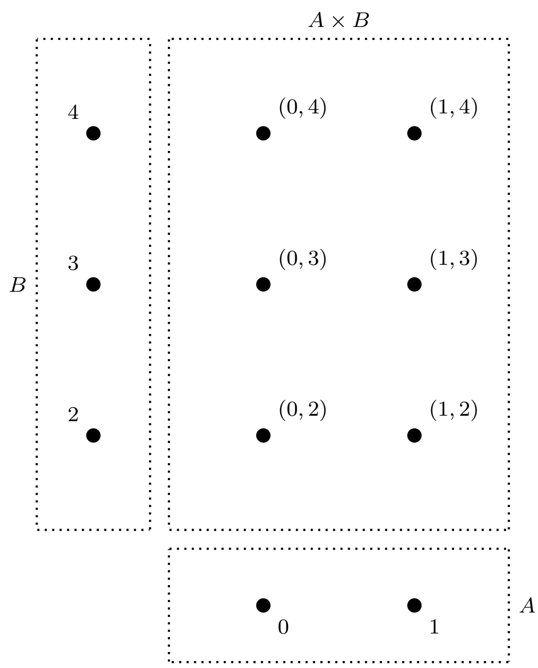
Remarks.

(1) According to convention on notations, we may simply write $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$.

(2) When $A = B$, we write $A \times B$ as A^2.

Examples.

(a) $\{0, 1\} \times \{2, 3, 4\} = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$.



(b) $\mathbb{R} \times \mathbb{R}$ is the 'coordinate plane' $\mathbb{R}^2$ in school mathematics.

4. **Ordered pairs as set-theoretic objects**.

   Philosophical question. *How to 'make sense' of the notion of ordered pairs in set language, in terms of objects already introduced in set language?*

   Definition.

   Let x, y be objects. The **Kuratowski ordered pair** *of x, y, with x being the first coordinate and y being the second coordinate, is defined to be the set $\{\{x\}, \{x, y\}\}$, and is denoted by $(x, y)_K$.*

   That this definition is an appropriate one is justified by the validity of Lemma (OP) below:

   Lemma (OP).

   Let x, y, u, v be objects. $(x, y)_K = (u, v)_K$ iff ($x = u$ and $y = v$).

   Proof. Exercise.

   Remark. From now on, we write $(x, y)_K$ as (x, y).

   Further remark. Is there another version of definition for the notion of ordered pairs?

   Wiener's version: $(x, y)_W = \{\{\emptyset, \{x\}\}, \{\{y\}\}\}$.

5. **Ordered triples and beyond**.

   We define the notion for ordered triples in terms of ordered pairs.

   Definition.

   Let x, y, z be objects. We define the **ordered triple** *of x, y, z, with first, second, third coordinates being x, y, z respectively, to be $((x, y), z)$. We write it as (x, y, z).*

   That this definition is an appropriate one is justified by the validity of Lemma (OT) below:

   Lemma (OT).

   Let x, y, z, u, v, w be objects. $(x, y, z) = (u, v, w)$ iff ($x = u$ and $y = v$ and $z = w$).

   Proof. Exercise.

   Remark. We may extend the idea in the definition for the notion of ordered triple so as to give the definition for the notions of ordered quadruples, ordered quintuples et cetera.

6. **Theorem ($*$). (Set-theoretic properties of cartesian products.)**

Let A, B, C, D be sets. The following statements hold:

(1) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.
$(A \cap B) \times (C \cap D) = (A \times D) \cap (B \times C)$.

(2) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
$A \times (C \cup D) = (A \times C) \cup (A \times D)$.

(3) *Suppose $A \subset B$ and $C \subset D$. Then $A \times C \subset B \times D$.*

(4) *Suppose $A \neq \emptyset$, $A \subset B$ and $A \times C \subset B \times D$. Then $C \subset D$.*

(5) $A \times \emptyset = \emptyset$. $\quad \emptyset \times A = \emptyset$.

Proof of the first equality in Statement (1) of Theorem ($*$).

Let A, B, C, D be sets.

- Pick any object t. Suppose $t \in (A \cap B) \times (C \cap D)$.
  By the definition of Cartesian product, there exist some $x \in A \cap B$, $y \in C \cap D$ such that $t = (x, y)$.
  In particular, $x \in A \cap B$. Since $A \cap B \subset A$, we have $x \in A$.
  Similarly, $y \in C \cap D \subset C$.
  Now $x \in A$ and $y \in C$. Therefore $t = (x, y) \in A \times C$ by the definition of Cartesian product.
  Modifying the above argument, we also deduce that $t \in B \times D$.
  Now we have $t \in A \times C$ and $t \in B \times D$.
  Therefore $t \in (A \times C) \cap (B \times D)$ by the definition of intersection.
- Pick any object t. Suppose $t \in (A \times C) \cap (B \times D)$.
  Then $t \in A \times C$ and $t \in B \times D$ by the definition of intersection.
  In particular $t \in A \times C$.
  By the definition of Cartesian product, there exist some $x \in A$, $y \in C$ such that $t = (x, y)$.
  Recall that $t \in B \times D$ also. There exist some $x' \in B$, $y' \in D$ such that $t = (x', y')$.
  We have $(x, y) = t = (x', y')$. Then ($x = x'$ and $y = y'$). Therefore $x \in B$ and $y \in D$.
  Now we have $x \in A$ and $x \in B$. Therefore $x \in A \cap B$ by the definition of intersection.
  We also have $y \in C$ and $y \in D$. Then $y \in C \cap D$.
  Therefore $t = (x, y) \in (A \cap B) \times (C \cap D)$ by the definition of Cartesian product.

It follows that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

Proof of Statement (3) of Theorem ($*$).

Let A, B, C, D be sets. Suppose $A \subset B$ and $C \subset D$.

Pick any object t. Suppose $t \in A \times C$. Then there exist some $x \in A$, $y \in C$ such that $t = (x, y)$.

Since $x \in A$ and $A \subset B$, we have $x \in B$. Since $y \in C$ and $C \subset D$, we have $y \in D$.

Then $t = (x, y) \in B \times D$.

It follows that $A \times C \subset B \times D$.