

1. Recall Method of specification (for the construction of sets):

Suppose A is a set and $P(x)$ is a predicate with variable x .

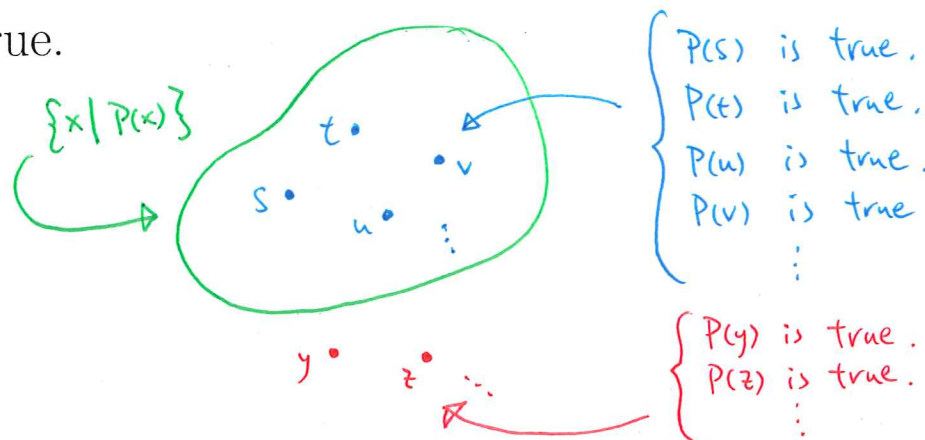
- $\{x \mid P(x)\}$ refers to the set (if it is indeed a set) which contains exactly every object x
 - * for which the statement $P(x)$ is true.

Always remember:

$$b \in \{x \mid P(x)\}$$

iff

$$P(b) \text{ is true.}$$



- $\{x \in A : P(x)\}$ refers to the set which contains exactly every object x
 - * which is an element of the given set A and
 - * for which the statement $P(x)$ is true.

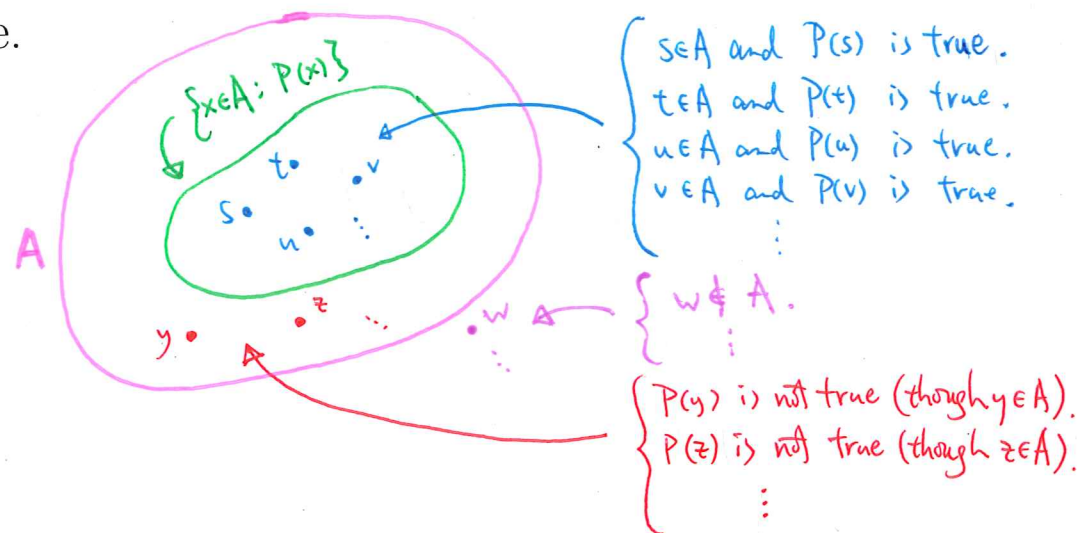
By definition it is a subset of A .

Always remember:

$$b \in \{x \in A : P(x)\}$$

iff

$$(b \in A \text{ and } P(b) \text{ is true.})$$



2. Recall the notions of **set equality** and ‘**subset relations**’:

- Let A, B be sets. A is said to be **equal** to B if both of the following statements (\dagger) , (\ddagger) hold:

(\dagger) For any object x , [if $(x \in A)$ then $(x \in B)$].

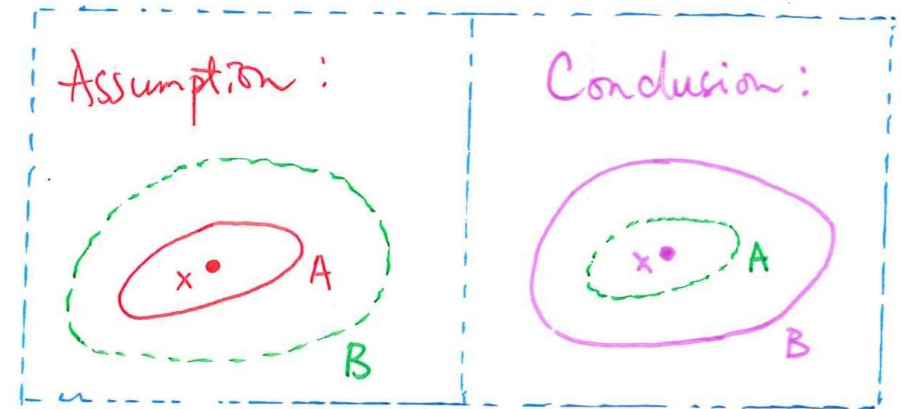
(\ddagger) For any object y , [if $(y \in B)$ then $(y \in A)$].

We write $A = B$.

- Let A, B be sets. A is said to be a **subset** of B if the following statement (\dagger) holds:

(\dagger) For any object x , [if $(x \in A)$ then $(x \in B)$]. ← For each object x , we have:

We write $A \subset B$ (or $B \supset A$).

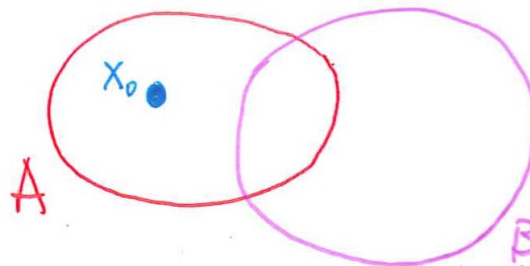


3. Question.

What do we mean by ‘ A is not a subset of B ’?

Answer.

A is not a subset of B exactly when some element of A fails to be an element of B .



More useful formulation for the same thing (though formal):

- *There exists some object x_0 such that $(x_0 \in A \text{ and } x_0 \notin B)$.*

In this situation, we write $A \not\subset B$.

4. **Example (a).**

Let $C = \{x \mid x = n^4 \text{ for some } n \in \mathbf{N}\}$, $D = \{x \mid x = n^2 \text{ for some } n \in \mathbf{N}\}$.

The statements below hold:

(1) $C \subset D$.

(2) $D \not\subset C$.

Heuristic ideas for the statements:

- C is the set of all biquadratic numbers while D is the set of all square numbers.
- Every biquadratic number is the square of a square number. So we expect ' $C \subset D$ ' to hold.
- There may be some square number which is not a biquadratic number; for instance, the square of a non-square number. So we expect ' $D \not\subset C$ ' to hold.

These are the core ideas in the proofs. They need be present before we write the proofs.

We organize these ideas to give a coherent argument, with reference to the definitions of C, D, \subset .

5. Proofs of the statements in Example (a).

(1) [We want to prove 'for any x , if $x \in C$ then $x \in D$ ']

Pick any object x . Suppose $x \in C$.

[What to deduce? ' $x \in D$ '. What does it read? 'Unwrap' ' $x \in D$ ' to see what it is.

How to reach ' $x \in D$ '? 'Unwrap' ' $x \in C$ ' to see what may help us.]

There exists some $m \in \mathbb{N}$
such that $x = m^2$.

Name an appropriate
 $m \in \mathbb{N}$
for which $x = m^2$

There exists some $n \in \mathbb{N}$
such that $x = n^4$.
This has been given
to us by assumption.

Since $x \in C$,

there exists some $n \in \mathbb{N}$ such that $x = n^4$.

Take $m = n^2$. Then $x = n^4 = (n^2)^2 = m^2$.

Since $n \in \mathbb{N}$, we have $m \in \mathbb{N}$.

Now we have $x = m^2$ and $m \in \mathbb{N}$.

Hence $x \in D$.

It follows that $C \subset D$. \square

Roughwork:
Any appropriate $m \in \mathbb{N}$
satisfying $x = m^2$?
For such an m ,
we expect
 $m^2 = x = n^4$.
So $m = n^2$.

Proofs of the statements in Example (a).

(2) [Preparation: find out what is to be done.]

We want to prove that there exists some x_0 such that $x_0 \in D$ and $x_0 \notin C$. (This is an existence statement.) So we look for an appropriate x_0 .

Does our heuristic understanding of C, D in this specific example help us spot a candidate? Is such a candidate a 'good one'?

[Roughwork: $C = \{0, 1, 16, 81, 625, \dots\}$
 $D = \{0, 1, 4, 9, 16, 25, \dots\}$
How about naming $x_0 = 4$?

Take $x_0 = 4$.

• [Ask: $x_0 \in D$?

Note that $x_0 = 2^2$ and $2 \in \mathbb{N}$.

Then $x_0 \in D$.

• [Ask: $x_0 \notin C$?

Claim: $x_0 \notin C$.

We justify this claim (with the help of proof-by-contradiction):

* Suppose it were true that $x_0 \in C$. Then there would exist some $n \in \mathbb{N}$ such that $x_0 = n^4$.

Now $4 = x_0 = n^4$.

Since $n \in \mathbb{R}$ and $n \geq 0$, we would have $n = \sqrt[4]{4}$. Now $n = \sqrt{2}$ and $n \in \mathbb{N}$. Contradiction arises. \square

7. **Example (e).**

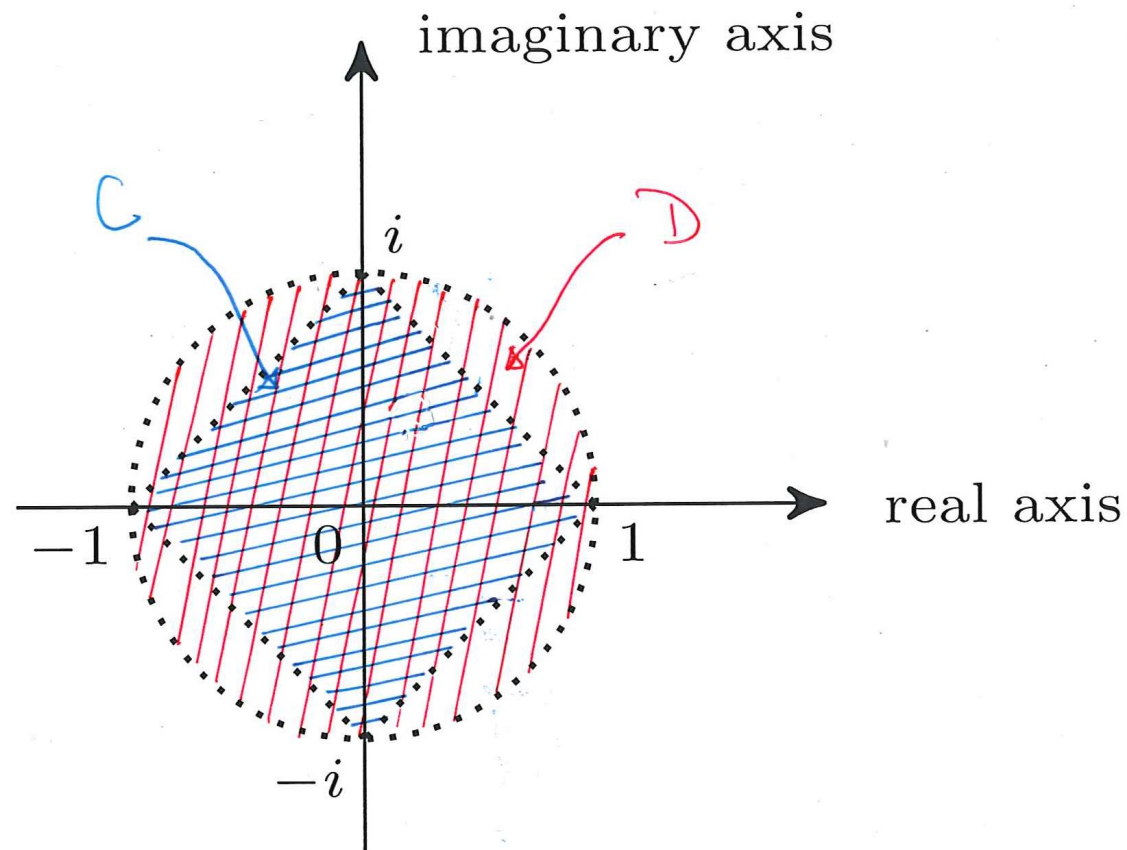
Let $C = \{\zeta \in \mathbb{C} : |\operatorname{Re}(\zeta)| + |\operatorname{Im}(\zeta)| < 1\}$, $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$.

The statements below hold:

(1) $C \subset D$.

(2) $D \not\subset C$.

Heuristic ideas for the statements, which can be visualized using the Argand plane:



8. Proofs of the statements in Example (e).

(1) [We want to prove 'for any $\zeta \in \mathbb{C}$, if $\zeta \in C$ then $\zeta \in D$ ']

Pick any object ζ . Suppose $\zeta \in C$.

[What to deduce? ' $\zeta \in D$ '. What does it read? ' $|\zeta| < 1$ '. How to reach ' $\zeta \in D$ '? Find out what ' $\zeta \in C$ ' reads: it is $|\operatorname{Re}(\zeta)| + |\operatorname{Im}(\zeta)| < 1$.]

Then $|\operatorname{Re}(\zeta)| + |\operatorname{Im}(\zeta)| < 1$. — (\star)

[How does $|\zeta|$ link up with $|\operatorname{Re}(\zeta)|$, $|\operatorname{Im}(\zeta)|$?]

We have

$$|\zeta|^2 = (\operatorname{Re}(\zeta))^2 + (\operatorname{Im}(\zeta))^2$$

$$= |\operatorname{Re}(\zeta)|^2 + |\operatorname{Im}(\zeta)|^2$$

$$= |\operatorname{Re}(\zeta)| \cdot |\operatorname{Re}(\zeta)| + |\operatorname{Im}(\zeta)| \cdot |\operatorname{Im}(\zeta)|$$

$$\leq |\operatorname{Re}(\zeta)| \cdot 1 + |\operatorname{Im}(\zeta)| \cdot 1$$

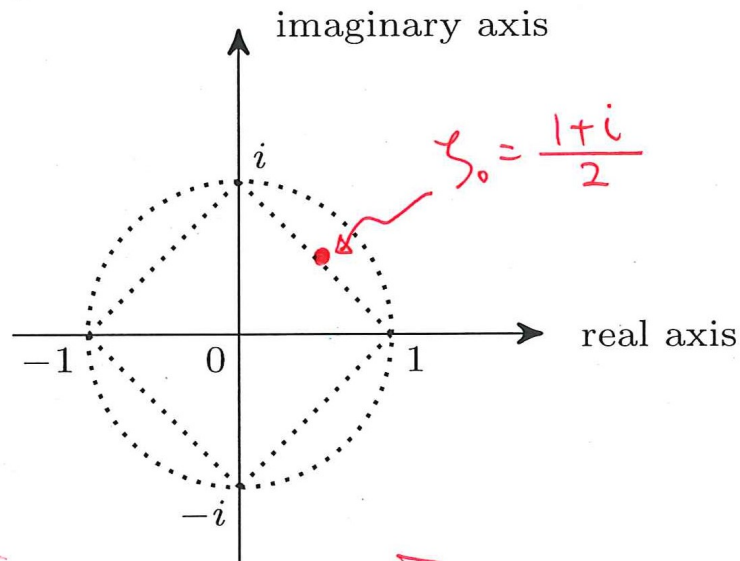
$$= |\operatorname{Re}(\zeta)| + |\operatorname{Im}(\zeta)| < 1 \quad \text{by } (\star).$$

Then $|\zeta| < 1$. Therefore $\zeta \in D$.

It follows that $C \subset D$.

Proofs of the statements in Example (e).

(2) [Preparation: find out what is to be done. We want to prove that there exists some ζ_0 such that $\zeta_0 \in D$ and $\zeta_0 \notin C$. (This is an existence statement.) So we look for an appropriate ζ_0 . \dots]



Ask:

- $\zeta_0 \in D$?
- $\zeta_0 \notin C$?

- Take $\zeta_0 = \frac{1+i}{2}$.
 - Note that
$$|\zeta_0|^2 = (\operatorname{Re}(\zeta_0))^2 + (\operatorname{Im}(\zeta_0))^2$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1.$$
Then $|\zeta_0| < 1$.
Therefore $\zeta_0 \in D$.
 - Note that
$$|\operatorname{Re}(\zeta_0)| + |\operatorname{Im}(\zeta_0)| = \frac{1}{2} + \frac{1}{2} = 1.$$
Then $\zeta_0 \notin C$.
- It follows that $D \not\subset C$.

9. Below are other examples similar to Example (e).

Example (f).

Let $C = \{\zeta \in \mathbf{C} : |\zeta - 1| \leq 1\}$, $D = \{\zeta \in \mathbf{C} : |\zeta| \leq 2\}$.

The statements below hold:

(1) $C \subset D$.

(2) $D \not\subset C$.

Example (g).

Let $C = \{\zeta \in \mathbf{C} : \operatorname{Re}(\zeta) \geq 0\}$, $D = \{\zeta \in \mathbf{C} : \operatorname{Im}(\zeta) \geq 0\}$, $E = \{\zeta \in \mathbf{C} : |\zeta - 1 - i| \leq 1\}$.

The statements below hold:

(1) $E \subset C \cap D$. (2) $C \not\subset D$. (3) $D \not\subset C$.

Example (h).

Let $C = \{\zeta \in \mathbf{C} : |\zeta - 4| < 5\}$, $D = \{\zeta \in \mathbf{C} : |\zeta + 4| < 5\}$, $E = \{\zeta \in \mathbf{C} : |\zeta| < 3\}$.

The statements below hold:

(1) $C \not\subset D$. (2) $D \not\subset C$. (3) $E \not\subset C$. (4) $E \not\subset D$. (5) $E \subset C \cup D$.

10. Example (i).

Let G be an $(m \times n)$ -matrix with real entries, and H be an $(n \times p)$ -matrix with real entries. The statements below hold:

- (1) The null space of H is a subset of the null space of GH .
- (2) Suppose the null space of G is $\{\mathbf{0}_n\}$. Then the null space of GH is a subset of the null space of H .

Remark. The null space $\mathcal{N}(K)$ of a $(p \times q)$ -matrix K with real entries is defined by $\mathcal{N}(K) = \{\mathbf{v} \in \mathbb{R}^q : K\mathbf{v} = \mathbf{0}_p\}$.

11. Proofs of the statements in Example (i).

- (1) [We want to prove ‘for any $\mathbf{x} \in \mathbb{R}^p$, if $\mathbf{x} \in \mathcal{N}(H)$ then $\mathbf{x} \in \mathcal{N}(GH)$ ’.]

Pick any $\mathbf{x} \in \mathbb{R}^p$. Suppose $\mathbf{x} \in \mathcal{N}(H)$.

[What to deduce? ‘ $\mathbf{x} \in \mathcal{N}(GH)$ ’. What does it read? ‘ $(GH)\mathbf{x} = \mathbf{0}_m$ ’. How to reach ‘ $(GH)\mathbf{x} = \mathbf{0}_m$ ’? Find out what ‘ $\mathbf{x} \in \mathcal{N}(H)$ ’ reads: it is ‘ $H\mathbf{x} = \mathbf{0}_n$ ’.]

Then by the definition of $\mathcal{N}(H)$, we have $H\mathbf{x} = \mathbf{0}_n$.

Therefore $(GH)\mathbf{x} = G(H\mathbf{x}) = G\mathbf{0}_n = \mathbf{0}_m$.

Hence, by the definition of $\mathcal{N}(GH)$, we have $\mathbf{x} \in \mathcal{N}(GH)$.

It follows that $\mathcal{N}(H) \subset \mathcal{N}(GH)$.

(2) Suppose the null space of G is $\{\mathbf{0}_n\}$.

[We want to deduce, under the above assumption, that ' $\mathcal{N}(GH) \subset \mathcal{N}(H)$ ', which reads: 'for any $\mathbf{x} \in \mathbb{R}^p$, if $\mathbf{x} \in \mathcal{N}(GH)$ then $\mathbf{x} \in \mathcal{N}(H)$ ']

Pick any $\mathbf{u} \in \mathbb{R}^p$. Suppose $\mathbf{u} \in \mathcal{N}(GH)$.

[What to deduce? ' $\mathbf{u} \in \mathcal{N}(H)$ '. What does it read? ' $H\mathbf{u} = \mathbf{0}_n$ '. How to reach ' $H\mathbf{u} = \mathbf{0}_n$ '? Find out what ' $\mathbf{u} \in \mathcal{N}(GH)$ ' reads: it is ' $(GH)\mathbf{u} = \mathbf{0}_m$ ']

Then by the definition of $\mathcal{N}(GH)$, we have $G(H\mathbf{u}) = (GH)\mathbf{u} = \mathbf{0}_m$.

Therefore, by the definition of $\mathcal{N}(G)$, we have $H\mathbf{u} \in \mathcal{N}(G)$.

Since $\mathcal{N}(G) = \{\mathbf{0}_n\}$, we have $H\mathbf{u} \in \{\mathbf{0}_n\}$. Then $H\mathbf{u} = \mathbf{0}_n$.

Therefore, by the definition of $\mathcal{N}(H)$, we have $\mathbf{u} \in \mathcal{N}(H)$.

It follows that $\mathcal{N}(GH) \subset \mathcal{N}(H)$.

12. Example (j).

Let S, T be subsets of \mathbb{R}^n , G be an $(m \times n)$ -matrix with real entries, and H be an $(n \times p)$ -matrix with real entries.

Define

$$S' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x} \text{ for some } \mathbf{x} \in S\},$$

$$T' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x} \text{ for some } \mathbf{x} \in T\}.$$

Define

$$S^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u} \text{ for some } \mathbf{x} \in S\},$$

$$T^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u} \text{ for some } \mathbf{x} \in T\}.$$

The statements below hold:

- (1) Suppose S is a subset of T . Then S' is a subset of T' .
- (2) Suppose S is a subset of T . Then S^* is a subset of T^* .

13. Proofs of the statements in Example (j).

(1) Suppose S is a subset of T .

[We want to deduce, under the above assumption, that ' S' is a subset of T' ', which reads: 'for any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in S'$ then $\mathbf{y} \in T'$ '.]

[Recall what S' and T' are:

$$S' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x} \text{ for some } \mathbf{x} \in S\},$$

$$T' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x} \text{ for some } \mathbf{x} \in T\}, \text{ in which } G \text{ is some fixed } (m \times n)\text{-matrix.}]$$

Pick any object $\mathbf{y} \in \mathbb{R}^m$. Suppose $\mathbf{y} \in S'$.

[What to deduce? ' $\mathbf{y} \in T'$ '. What does it read? 'Unwrap' ' $\mathbf{y} \in T'$ ' to see what it is. How to reach ' $\mathbf{y} \in T'$ '? 'Unwrap' ' $\mathbf{y} \in S'$ ' to see what may help us.]

Then by the definition of S' , there exists some $\mathbf{x} \in S$ such that $\mathbf{y} = G\mathbf{x}$.

Note that $\mathbf{x} \in S$, and by assumption S is a subset of T . Then, by the definition of subset relations, $\mathbf{x} \in T$.

Therefore $\mathbf{x} \in T$ and $\mathbf{y} = G\mathbf{x}$ for the same \mathbf{x}, \mathbf{y} .

Hence, by the definition of T' , we have $\mathbf{y} \in T'$.

It follows that $S' \subset T'$.

(2) Suppose S is a subset of T .

[We want to deduce, under the above assumption, that that ‘ S^* is a subset of T^* ’, which reads: ‘for any $\mathbf{u} \in \mathbb{R}^p$, if $\mathbf{u} \in S^*$ then $\mathbf{u} \in T^*$ ’.]

[Recall what S^* and T^* are:

$$S^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u} \text{ for some } \mathbf{x} \in S\},$$

$$T^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u} \text{ for some } \mathbf{x} \in T\}, \text{ in which } H \text{ is some fixed } (m \times p)\text{-matrix.}]$$

Pick any object $\mathbf{u} \in \mathbb{R}^p$. Suppose $\mathbf{u} \in S^*$.

[What to deduce? ‘ $\mathbf{u} \in T^*$ ’. What does it read? ‘Unwrap’ ‘ $\mathbf{u} \in T^*$ ’ to see what it is. How to reach ‘ $\mathbf{u} \in T^*$ ’? ‘Unwrap’ ‘ $\mathbf{u} \in S^*$ ’ to see what may help us.]

Then by the definition of S^* , there exists some $\mathbf{x} \in S$ such that $\mathbf{x} = H\mathbf{u}$.

Note that $\mathbf{x} \in S$, and by assumption S is a subset of T . Then, by the definition of subset relations, $\mathbf{x} \in T$.

Therefore $\mathbf{x} \in T$ and $\mathbf{x} = H\mathbf{u}$, for the same \mathbf{x}, \mathbf{u} .

Hence, by the definition of T^* , we have $\mathbf{u} \in T^*$.

It follows that $S^* \subset T^*$.