1. **Definition.**

*Let $z$ be a complex number.*

*The* **modulus** *of $z$, which we denote by $|z|$, is defined by $|z| = \sqrt{(\mathsf{Re}(z))^2 + (\mathsf{Im}(z))^2}$.*

*The expression $z = |z|(\cos(\theta) + i\sin(\theta))$ (for some appropriate real number $\theta$) is called the* **polar form** *of $z$.*

*If $z \neq 0$, then such a number $\theta$ is called an* **argument** *for $z$. Furthermore, if $-\pi < \theta \leq \pi$, then $\theta$ is called the* **principal argument** *of $z$.*

**Remark.** This definition makes sense is guaranteed by the statement below, which needs be justified carefully:

- *Let $z$ be a complex number. There exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta) + i\sin(\theta))$.*

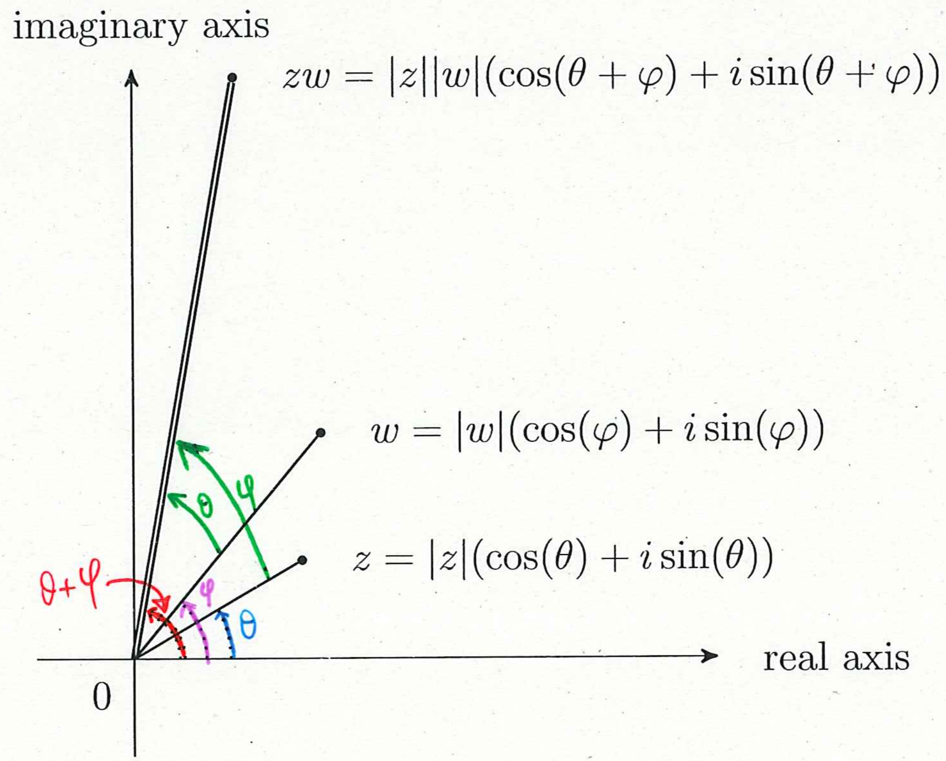
**Further remark.** Multiplication and division for complex numbers can be given a nice geometric interpretation in terms of polar form:

*Suppose $z, w$ are non-zero complex numbers, with arguments $\theta, \varphi$ respectively. Then:*

(a) $zw = |z||w|(\cos(\theta+\varphi) + i\sin(\theta+\varphi))$, *and ...*

(b) *The modulus of $zw$ is $|z||w|$, and ...*

(c) $\theta+\varphi$ *is an argument for $zw$, and ...*



2. **Lemma (1). (Special case of De Moivre's Theorem.)**

*Let $\theta$ be a real number. For any $n \in \mathbb{N}\backslash\{0\}$, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.*

**Proof.** Let $\theta$ be a real number.

- For any $n \in \mathbb{N}\backslash\{0\}$, denote by $P(n)$ the proposition
$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$
- $(\cos(\theta) + i\sin(\theta))^1 = \cos(1 \cdot \theta) + i\sin(1 \cdot \theta)$. Then $P(1)$ is true.
- Let $k \in \mathbb{N}\backslash\{0\}$. Suppose $P(k)$ is true. Then $(\cos(\theta) + i\sin(\theta))^k = \cos(k\theta) + i\sin(k\theta)$.
We prove that $P(k+1)$ is true:
$$\begin{aligned}
& (\cos(\theta) + i\sin(\theta))^{k+1} \\
= \ & (\cos(\theta) + i\sin(\theta))^k(\cos(\theta) + i\sin(\theta)) \\
= \ & (\cos(k\theta) + i\sin(k\theta))(\cos(\theta) + i\sin(\theta)) \\
= \ & (\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)) + i(\sin(k\theta)\cos(\theta) + \cos(k\theta)\sin(\theta)) \\
= \ & \cos(k\theta + \theta) + i\sin(k\theta + \theta) = \cos((k+1)\theta) + i\sin((k+1)\theta)
\end{aligned}$$
Hence $P(k+1)$ is true.
- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0\}$.

3. **De Moivre's Theorem.**

*Let $\theta$ be a real number. For any $m \in \mathbb{Z}$, $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.*

**Proof.** Let $\theta$ be a real number. Let $m \in \mathbb{Z}$.

- (Case 1). Suppose $m = 0$. Then

  $$(\cos(\theta)+i\sin(\theta))^m = (\cos(\theta)+i\sin(\theta))^0 = 1 = (\cos(0\cdot\theta)+i\sin(0\cdot\theta)) = \cos(m\theta)+i\sin(m\theta).$$

- (Case 2). Suppose $m > 0$.

  By Lemma (1), we have $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

- (Case 3). Suppose $m < 0$. Define $n = -m$. Then $n \in \mathbb{N}\backslash\{0\}$. Therefore

$$\begin{aligned}
(\cos(\theta) + i\sin(\theta))^m &= \frac{1}{(\cos(\theta) + i\sin(\theta))^n} \\
&\overset{\text{By Lemma(1)}}{=} \frac{1}{\cos(n\theta) + i\sin(n\theta)} \\
&\overset{\text{By `}z\bar{z}=|z|^2\text{'}}{=} \cos(n\theta) - i\sin(n\theta) \\
&= \cos(m\theta) + i\sin(m\theta).
\end{aligned}$$

Hence in any case, $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

4. **Definition.**

*Let $\zeta$ be a complex number. Let $n$ be a positive integer. $\zeta$ is called an* **$n$-th root of unity** *if $\zeta^n = 1$.*

**Remark.** $\zeta$ is an $n$-th root of unity iff $\zeta$ is a root of the polynomial $z^n - 1$ in the complex numbers.)

5. **Theorem (2).**

*Let $n$ be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.*

(a) *$\omega_n$ is an $n$-th root of unity.*

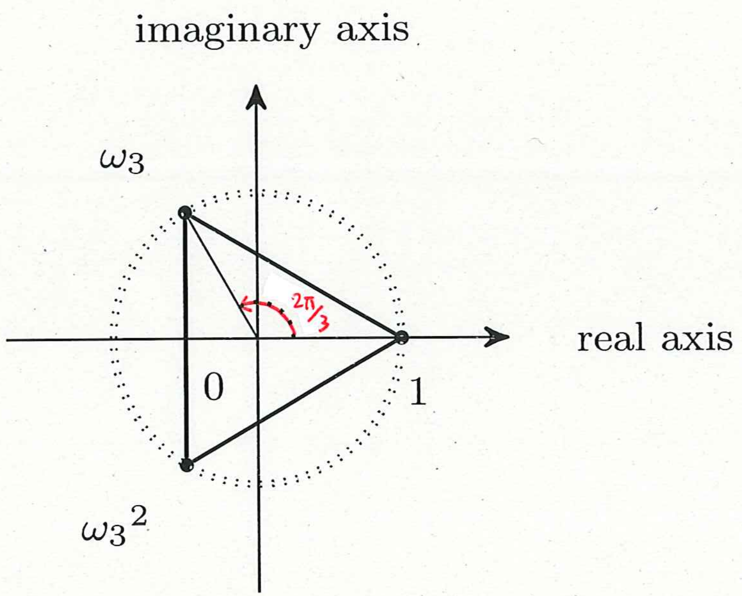
(b) *The $n$-th roots of unity are the $n$ complex numbers of modulus $1$, given by*

$$1, \omega_n, {\omega_n}^2, ..., {\omega_n}^{n-1}.$$

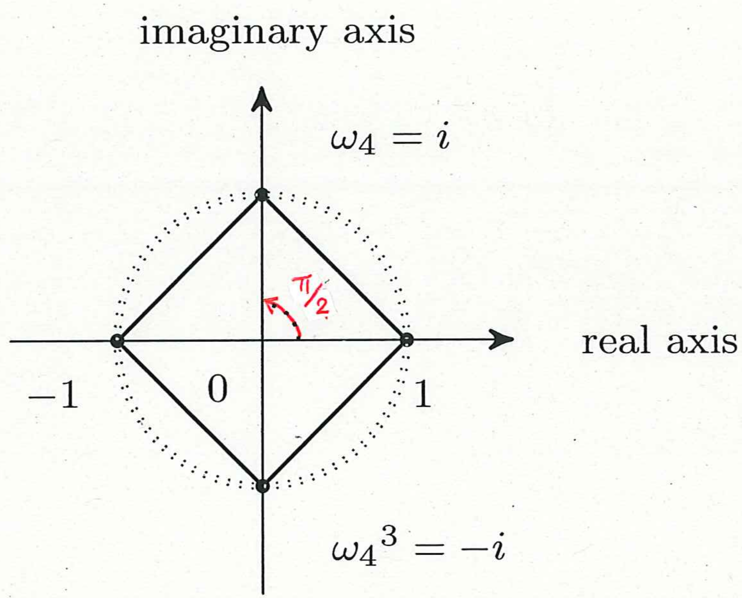
**Remark to Theorem (2).** How to visualize these $n$ numbers in terms of plane geometry?

They are the $n$ vertices of the regular $n$-sided polygon inscribed in the unit circle with centre 0 in the Argand plane, with one vertex at the point 1.

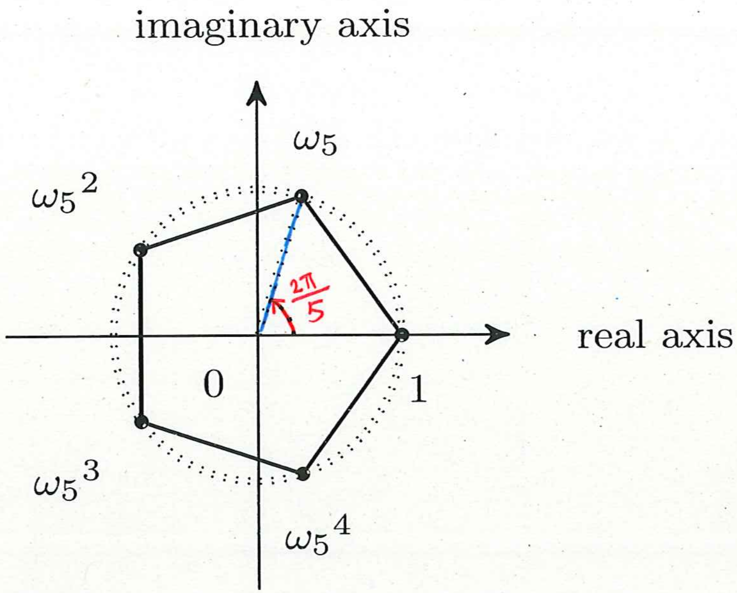
$n = 3$:




$n = 4$:

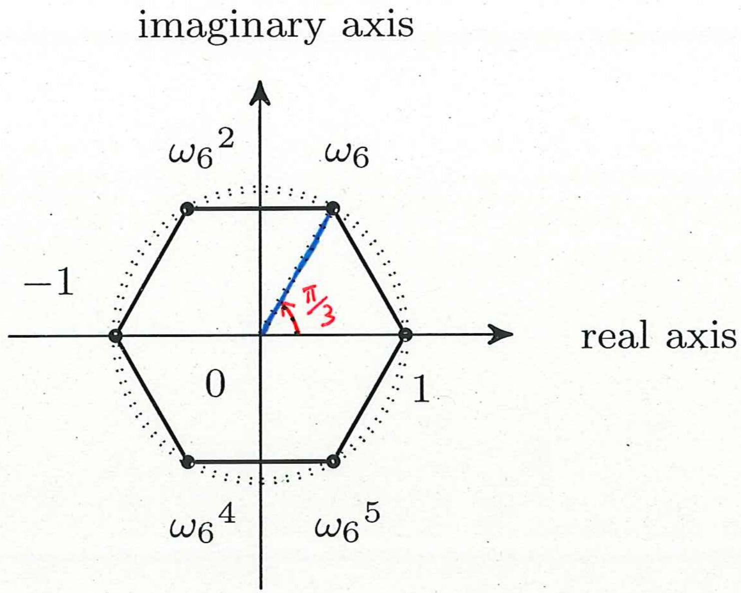


$n = 5$:



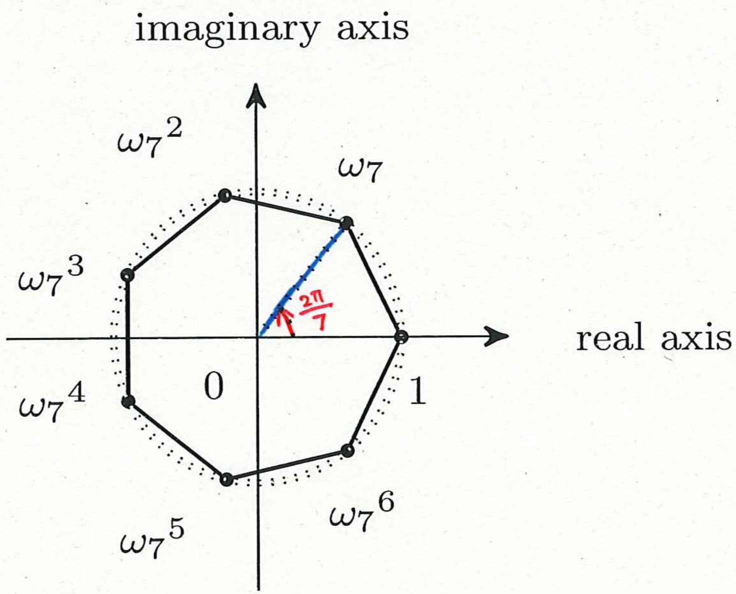
imaginary axis
real axis
$\omega_5$
${\omega_5}^2$
${\omega_5}^3$
${\omega_5}^4$
$0$
$1$
$\frac{2\pi}{5}$


$n = 6$:



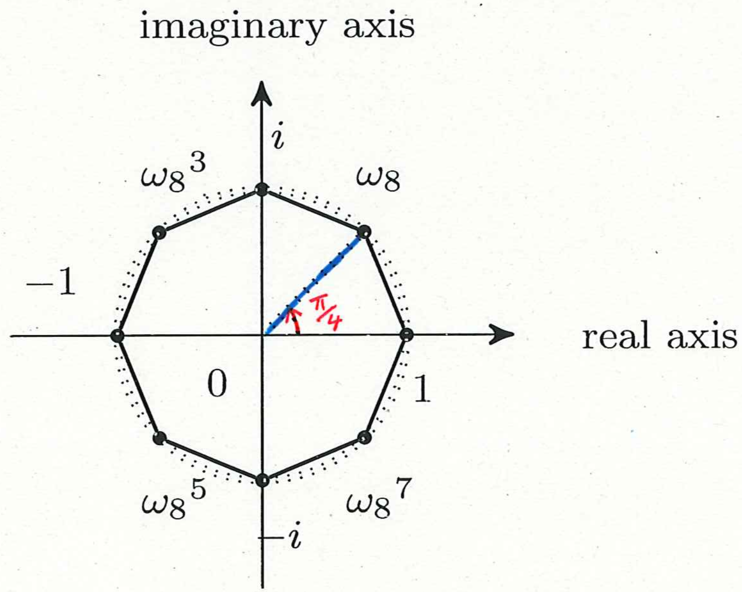
imaginary axis
real axis
$\omega_6$
${\omega_6}^2$
${\omega_6}^4$
${\omega_6}^5$
$-1$
$0$
$1$
$\pi/3$


$n = 7$:



imaginary axis
real axis
$\omega_7$
${\omega_7}^2$
${\omega_7}^3$
${\omega_7}^4$
${\omega_7}^5$
${\omega_7}^6$
$0$
$1$
$\frac{2\pi}{7}$


$n = 8$:



imaginary axis
real axis
$i$
$-i$
$\omega_8$
${\omega_8}^3$
${\omega_8}^5$
${\omega_8}^7$
$-1$
$0$
$1$
$\pi/4$

6. **Tacit assumption need in the argument for Theorem (2).**

A tacit assumption, known as **Division Algorithm for integers**, is used in the argument. It reads:

> *Let* $u, v \in \mathbb{Z}$. *Suppose* $v \neq 0$. *Then there exist some unique* $q, r \in \mathbb{Z}$ *such that* $u = qv + r$ *and* $0 \leq r < |v|$.

## 7. Proof of Theorem (2).

Tacitly assumed result to be applied at (✱):
Let $u, v \in \mathbb{Z}$. Suppose $v \neq 0$. Then there exist some unique $q, r \in \mathbb{Z}$ such that $u = qv + r$ and $0 \leq |r| < v$.

Let $n$ be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

(a) By De Moivre's Theorem, $(\omega_n)^n = (\cos(n\theta_n) + i\sin(n\theta_n)) = \cos(2\pi) + i\sin(2\pi) = 1$.

(b) i. For each $k = 0, 1, 2, \cdots, n-1$, we have $(\omega_n{}^k)^n = (\omega_n{}^n)^k = 1^k = 1$.

ii. Let $\zeta$ be a complex number. Suppose $\zeta$ is an n-th root of unity. Then $\zeta^n = 1$.

[We want to deduce that $\zeta = \omega_n{}^r$ for some $r \in [\![0, n-1]\!]$.]

We have $|\zeta|^n = |\zeta^n| = 1$. Then $|\zeta| = 1$.

$\zeta$ has an argument, say, $\varphi$. Therefore $\zeta = \cos(\varphi) + i\sin(\varphi)$.

[Ask: what more can we say about $\varphi$?]

By De Moivre's Theorem, $1 = \zeta^n = (\cos(\varphi) + i\sin(\varphi))^n = \cos(n\varphi) + i\sin(n\varphi)$.

Then $\cos(n\varphi) = 1$ and $\sin(n\varphi) = 0$.

Therefore there exists some $m \in \mathbb{Z}$ such that $n\varphi = 2m\pi$.

Now $\varphi = \dfrac{m}{n} \cdot 2\pi = m\theta_n$

(✱) ⟿ By Division Algorithm, there exist some $q, r \in \mathbb{Z}$ such that $m = qn + r$ and $0 \leq r < n$.

Then $\varphi = m\theta_n = (qn + r)\theta_n = qn\theta_n + r\theta_n = 2q\pi + r\theta_n$

Therefore $\zeta = \cos(\varphi) + i\sin(\varphi) = \cos(r\theta_n) + i\sin(r\theta_n) = \omega_n{}^r$. □

8. **Corollary (3).**

*Let $n$ be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.*

*The polynomial $z^n - 1$ with indeterminate $z$ is completely factorized as*

$$z^n - 1 = (z-1)(z-\omega_n)(z-{\omega_n}^2)\cdot \ldots \cdot (z-{\omega_n}^{n-1}).$$

**Proof.** Exercise. (Apply Factor Theorem.)

**Remark.** In fact, the polynomial $z^n - 1$ can be factorized as a product of finitely many quadratic polynomials with real coefficients

$$z^2 - 2z\cos(\theta_n) + 1, \quad z^2 - 2z\cos(2\theta_n) + 1, \quad z^2 - 2z\cos(3\theta_n) + 1, \cdots$$

and the linear polynomial $z-1$ and, when $n$ is even, also together with the linear polynomial $z+1$. (The argument starts with the observation that ${\omega_n}^{-1} = \overline{\omega_n}$. Why? How?)

9. **Definition.**

*Let $n$ be a positive integer. Let $w, \zeta$ be complex numbers. $\zeta$ is said to be an **$n$-th root** of $w$ if $\zeta^n = w$.*

**Remark.** $\zeta$ is an $n$-th root of $w$ iff $\zeta$ is a root of the polynomial $z^n - w$ in the complex numbers.

10. **Lemma (4).**

*Let $n$ be a positive integer. Let $w$ be a non-zero complex number. Suppose $\varphi$ is an argument for $w$.*

*Then $\zeta = \sqrt[n]{|w|}(\cos(\varphi/n) + i\sin(\varphi/n))$ is an $n$-th root of $w$.*

**Proof.** Exercise. (Apply De Moivre's Theorem.)

11. **Theorem (5).**

*Let $n$ be a positive integer. Write $\theta_n = 2\pi/n$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.*

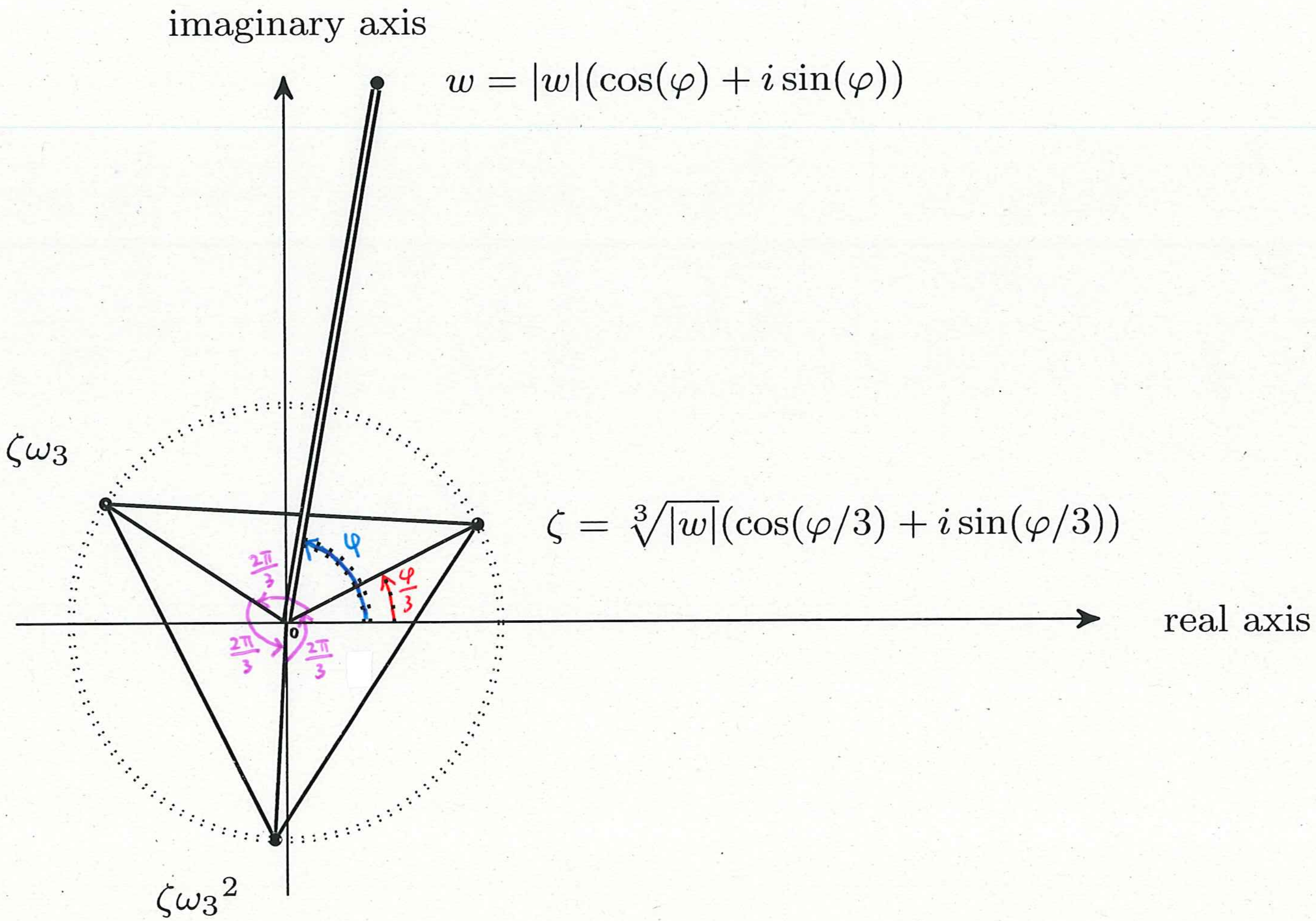
*Let $w$ be a non-zero complex number, and $\zeta$ be an $n$-th root of $w$ in the complex numbers.*

*The $n$-th roots of $w$ are the $n$ complex numbers given by $\zeta, \zeta\omega_n, \zeta{\omega_n}^2, \cdots, \zeta{\omega_n}^{n-1}$.*

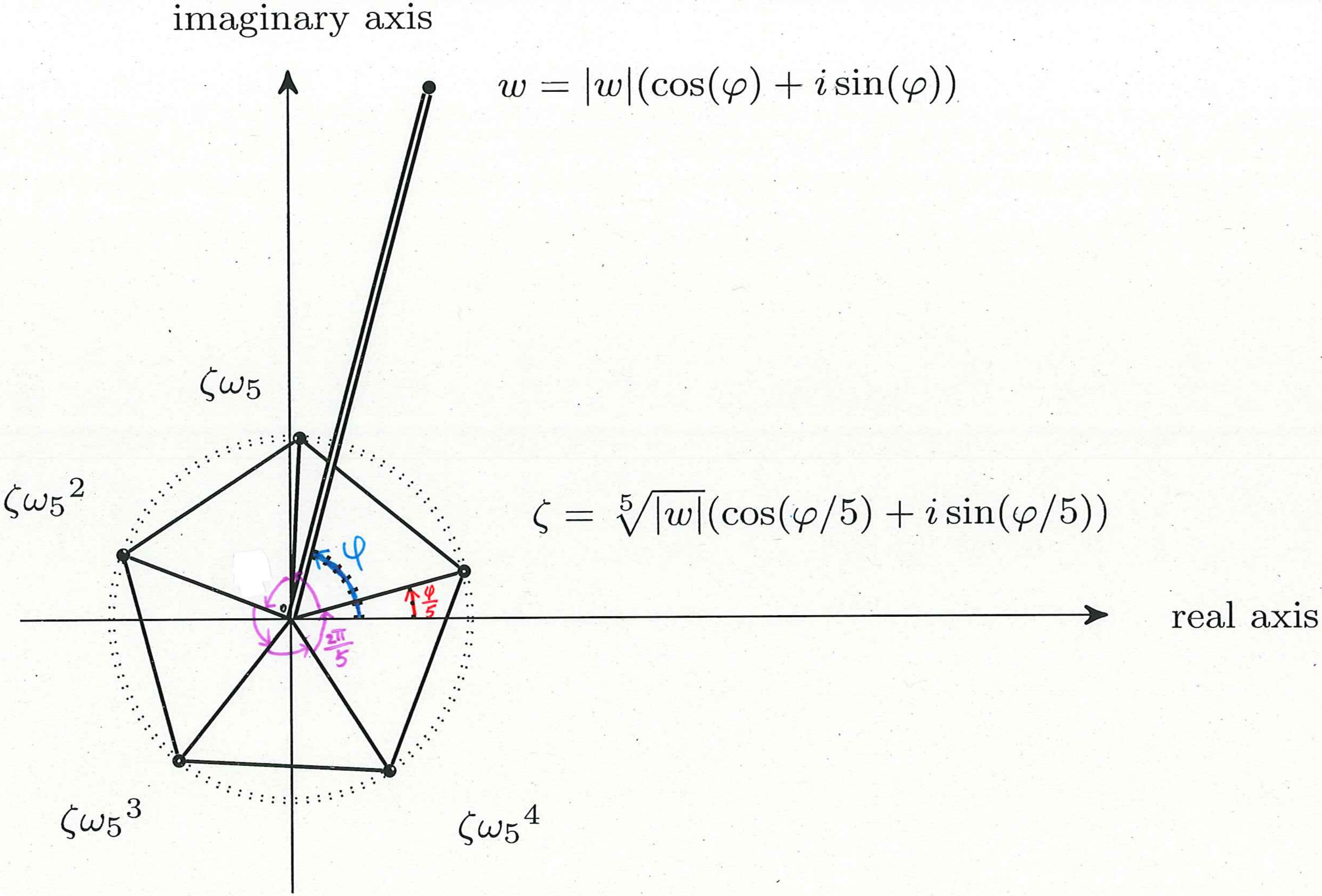
**Remark.** How to visualize these $n$ numbers in terms of plane geometry?

They are the $n$ vertices of the regular $n$-sided polygon inscribed in the circle with centre 0 and radius $\sqrt[n]{|w|}$ in the Argand plane, with one vertex at the point $\zeta$.

- Cubic roots:



- Quintic roots:



12. **Proof of Theorem (5).**

Let $n$ be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

Let $w$ be a non-zero complex number, and $\zeta$ be an $n$-th root of $w$ in the complex numbers.

- We have $\zeta^n = w$.

  For each $n = 0, 1, 2, \cdots, n-1$, we have $({\omega_n}^k)^n = 1$.

  Then $(\zeta{\omega_n}^k)^n = \zeta^n({\omega_n}^n)^k = 1 \cdot 1^k = 1$.

- Let $\rho$ be a complex number. Suppose $\rho$ is an $n$-th root of $w$.

  Then $\rho^n = w$. We have $\left(\dfrac{\rho}{\zeta}\right)^n = \dfrac{\rho^n}{\zeta^n} = \dfrac{w}{w} = 1$.

  Then $\dfrac{\rho}{\zeta}$ is an $n$-th root of unity.

  Therefore there exists some $r = 0, 1, 2, \cdots, n-1$ such that $\dfrac{\rho}{\zeta} = {\omega_n}^r$.

  For the same $r$, we have $\rho = \zeta{\omega_n}^r$.

13. **Corollary (6).**

*Let $n$ be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.*

*Let $w$ be a non-zero complex number, and $\zeta$ be an $n$-th root of $w$ in the complex numbers.*

*The polynomial $z^n - w$ with indeterminate $z$ is completely factorized as*

$$z^n - w = (z - \zeta)(z - \zeta\omega_n)(z - \zeta{\omega_n}^2) \cdot \ldots \cdot (z - \zeta{\omega_n}^{n-1}).$$

**Proof.** Exercise. (Apply Factor Theorem.)