1. **Undefined notions: 'set', 'belong to', 'element'.**

Unexplained statements which have the same meaning:

- 'x belongs to the set A'.
- 'x is an element of A'.
- 'A contains x as an element'.

At this level we allow heuristics to take over.

Short-hand for 'x is an element of A':

$$\text{`}x \in A\text{'}.$$

Short-hand for 'x is not an element of A':

$$\text{`}x \notin A\text{'}.$$

Everything else in set language is defined in terms of these notions.

2. **Heuristic understanding of the notions of 'set equality', 'subset relation'.**

(a) Any two sets A, B are **equal** to each other as sets exactly when:

- each of A, B contains as its elements every element of the other.

We write

$$A = B.$$

(b) Given any two sets A, B, A is a **subset** of B exactly when:

- every element of A is an element of B. ← More clumsy, but more useful formulation:
'For any object x, if $x \in A$ then $x \in B$.'

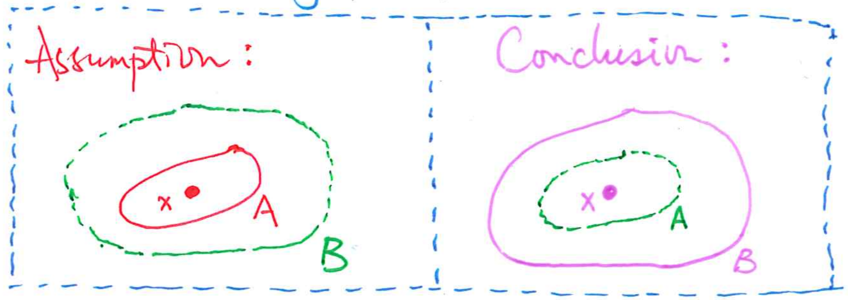
In this situation we write

$$A \subset B.$$

Visualization of definition:

For each object x, we have:

Some people's convention: '$A \subseteq B$'.



Reminder: When $A = B$ holds, it will happen that $A \subset B$ also holds.

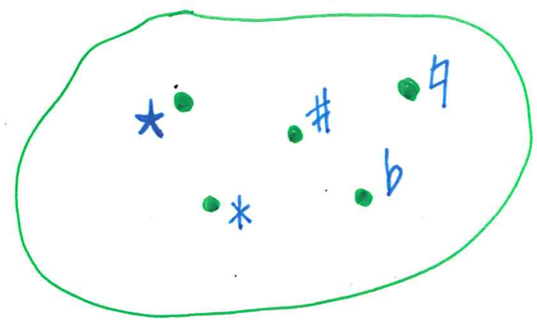
3. **‘Small’ sets.**

When a set has ‘finitely many’ elements, we may list every one of them exhaustively:

- The symbol ‘$\{$’ signifies the beginning of the list of elements.
- The symbol ‘$\}$’ signifies the end of the list of elements.

Example.

Venn diagram for this set:



- Suppose $\star, *, \sharp, \flat, \natural$ are the only elements of a set.
  This set may be expressed as $\{\star, *, \sharp, \flat, \natural\}$.
  Never write it as $(\star, *, \sharp, \flat, \natural)$, $[\star, *, \sharp, \flat, \natural]$.

Conventions.

1. ‘Repetition in the list’ does not count.

   Example:

   - $\{\sharp, \sharp, \flat, \natural, \flat, \flat\} = \{\sharp, \flat, \natural\}$.

2. ‘Ordering in the list’ does not matter.

   Example:

   - $\{\sharp, \flat, \natural\} = \{\flat, \sharp, \natural\} = \{\natural, \flat, \sharp\}$.

The set which has no element is called the **empty set**. We denote this set by $\emptyset$.

4. **Method of Specification**.

Many a set cannot be presented as a list, because it is not 'small'.

Even though a set may be presented as an list, for one reason or other we may choose not to do so.

Examples:

1. Consider the collection

   $$\text{`}0, 1, 4, 9, 16, 25, 36, \cdots\text{'}.$$

   Is it apparent that it refers to the collection of all square integers?

   But why can't it be understood as the collection of $0, 1, 4, 9, 16, 25$ and the integers no less than 36?

2. Consider the collection

   $$\text{`}1, 2, 3, 4, \cdots; \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots; \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \cdots; ...\text{'}$$

   Is it apparent that it refers to the collection of all rational numbers? Or is it not?

   Can you conceive a better list than this one?

   Or is it desirable to describe the collection of all positive rational numbers in this way?

When it is impossible or undesirable to present a set by exhaustively listing every element of the set, we may try the **Method of Specification**.

In such a set presented witht the Method of Specification, its elements are:

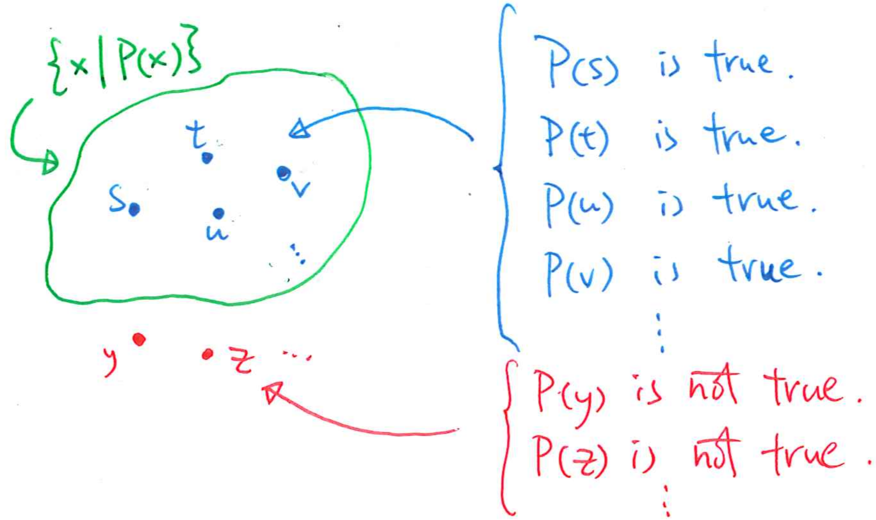
- those objects, and those alone, which turn a predicate 'used for describing that set' into a true statement.

Recall:

- A **predicate with variables** $x, y, z, \cdots$ is a statement 'modulo' the ambiguity of possibly one or several variables $x, y, z, \cdots$. Provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

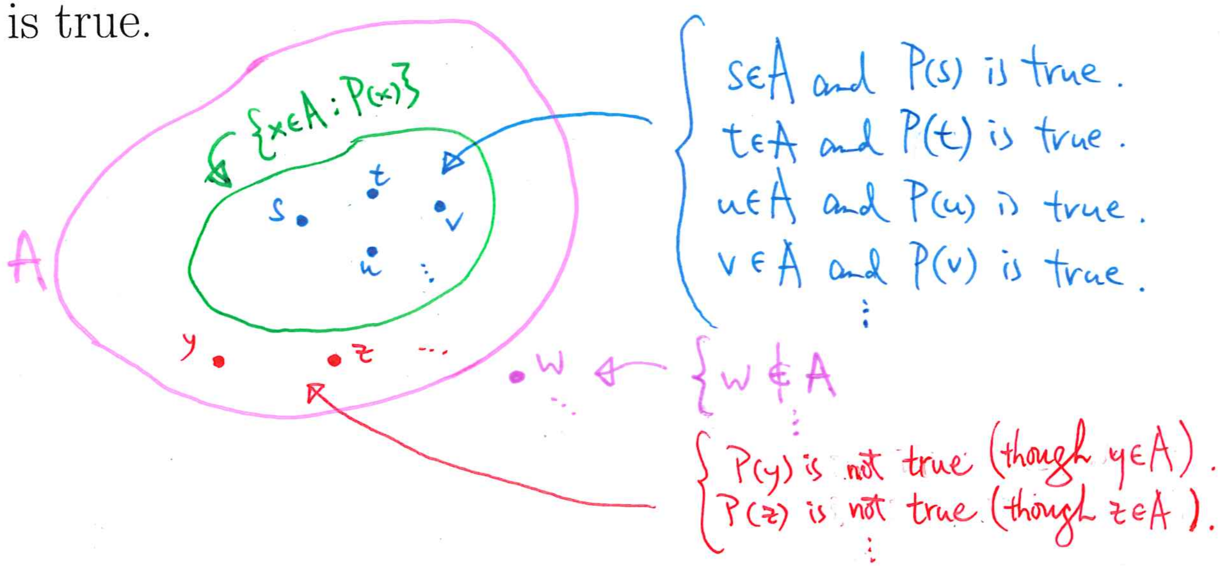
Suppose A is a set, and $P(x)$ is a predicate with variable x.

1. $\{x \mid P(x)\}$ refers to the set (if it is indeed a set) which contains exactly every object x
   - for which the statement $P(x)$ is true.



2. $\{x \in A : P(x)\}$ refers to the set which contains exactly every object x
   - which is an element of the given set A and
   - for which the statement $P(x)$ is true.

   By definition it is a subset of A.



5. **Examples on Method of Specification.**

(a) $\{x \mid x = \star \text{ or } x = * \text{ or } x = \sharp \text{ or } x = \flat \text{ or } x = \natural\} = \{\star, *, \sharp, \flat, \natural\}$ as sets.

This is a predicate with variable x.

(b) What is the set $\{x \in \mathbb{R} : x^2 - 3x + 2 = 0\}$ in plain words?

Solution set of the equation $x^2 - 3x + 2 = 0$ with unknown x in $\mathbb{R}$

(c) What are the sets below in plain words?

(i) $\{x \in \mathbb{R} : x^2 + 1 = 0\}$.

Solution set of the equation $x^2 + 1 = 0$ with unknown x in $\mathbb{R}$.

This is the empty set.

(ii) $\{x \in \mathbb{C} : x^2 + 1 = 0\}$.

Solution set of the equation $x^2 + 1 = 0$ with unknown x in $\mathbb{C}$.

This is $\{i, -i\}$.

Remark. But how about $\{x \mid x^2 + 1 = 0\}$? Suggestions: $\emptyset$, $\{i, -i\}$.

How about this? unknown 2x2-matrix with real entries. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(d) Heuristically describe the set $\{x \in \mathbb{R} : \sqrt{x} \in \mathbb{N}\}$.

$$\{x \in \mathbb{R} : \sqrt{x} \in \mathbb{N}\} \text{ `='} \{0, 1, 4, 9, 16, \cdots, n^2, (n+1)^2, \cdots\}.$$

We have eliminated the annoying dots.

How to see the answer? Ask:

- If $x \in \mathbb{R}$ and $\sqrt{x} \in \mathbb{N}$, what can x be?

Now collect all such x's.

Pragmatic method for seeing the elements of the set in (d'i).

(d') What about these sets below?

(i) $\{x \in \mathbb{R}$: There exists some $n \in \mathbb{N}$ such that $x = n^2\}$.

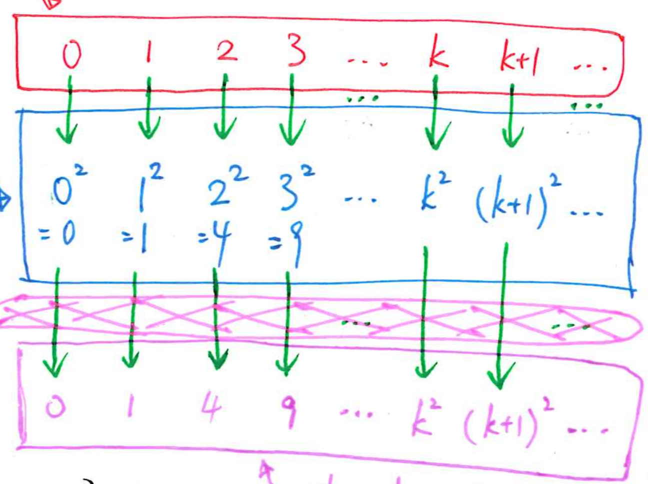
(ii) $\{x \in \mathbb{R} : x = n^2$ for some $n \in \mathbb{N}\}$.

(iii) $\{x \mid x = n^2$ for some $n \in \mathbb{N}\}$.

Various values of n in $\mathbb{N}$.

Various values of x as generated by n via the relation '$x = n^2$'.

'Sieve' defined by the condition '$x \in \mathbb{R}$'.



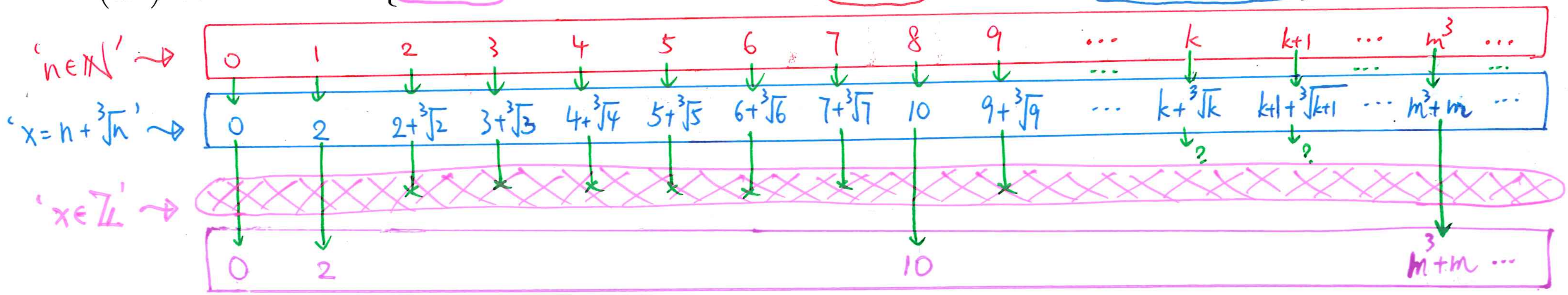
The elements of the set in (d'i)

Each of them is the same as $\{0, 1, 4, 9, 16, \cdots, n^2, (n+1)^2, \cdots\}$.

Because of (iii), we also accept this set to be expressed as $\{n^2 \mid n \in \mathbb{N}\}$.

$\{x \mid x = n^2$ for some $n \in \mathbb{N}\}$

(d”) What about $\{x \in \mathbb{Z} : \text{There exists some } n \in \mathbb{N} \text{ such that } x = n + \sqrt[3]{n}\}$?



This is the same as $\{0, 2, \cdots, m^3 + m, (m+1)^3 + m + 1, \cdots\}$.

(d”’) What about $\{x \in \mathbb{R} : x = n^2 \text{ for any } n \in \mathbb{N}\}$?

Inspect the predicate ‘$x = n^2$ for any $n \in \mathbb{N}$’. Ask:

- If the ‘substitution’ $x = x_0$ ‘turns’ this predicate into a true statement, what happens next?

Because $0 \in \mathbb{N}$, $x_0 = 0^2 = 0$.
Because $1 \in \mathbb{N}$, $x_0 = 1^2 = 1$.
Then $0 = x_0 = 1$. Contradiction arises.

This is the empty set.

(e) Heuristically describe the set $\{x \mid x = 3^m 5^n \text{ for some } m, n \in \mathbb{N}\}$.
This is the same as $\{1, 3, 9, 27, \cdots ; 5, 15, 45, \cdots ; 25, 75, 225, \cdots ; \ldots\}$.
Remark. We also accept this set to be expressed as $\{3^m 5^n \mid m \in \mathbb{N} \text{ and } n \in \mathbb{N}\}$.

(f) How to apply the Method of Specification to express $0, 3, 6, \cdots, 3n, 3n+3, \cdots$ as a set?

$$\begin{array}{ccccccc} 0 & 1 & 2 & \cdots & n & n+1 & \cdots \end{array}$$

There are many correct answers. Each of them refer to the same set.

- $\{3n \mid n \in \mathbb{N}\}$,
- $\{x \mid x = 3n \text{ for some } n \in \mathbb{N}\}$,
- $\{x \in \mathbb{Z} : x = 3n \text{ for some } n \in \mathbb{N}\}$,
- $\{x \in \mathbb{N} : x/3 \in \mathbb{N}\}$.

$$\begin{array}{ccccccccc} -6 & -3 & 0 & 3 & 6 & \cdots & 3n-6 & 3n-3 & \cdots \\ 3(0-2) & 3(1-2) & 3(2-2) & 3(3-2) & 3(4-2) & \cdots & 3(n-2) & 3(n-1) & \cdots \\ 0 & 1 & 2 & 3 & 4 & \cdots & n & n+1 & \cdots \end{array}$$

(f') How about the collection $-6, -3, 0, 3, 6, \cdots$?
There are many correct answers. Each of them refer to the same set.

- $\{3(n-2) \mid n \in \mathbb{N}\}$,
- $\{x \mid x = 3(n-2) \text{ for some } n \in \mathbb{N}\}$,
- $\{x \in \mathbb{Z} : x = 3(n-2) \text{ for some } n \in \mathbb{N}\}$,
- $\{x \in \mathbb{Z} : (x+6)/3 \in \mathbb{N}\}$.

(f") How about the collection $\cdots, -6, -3, 0, 3, 6, \cdots$?
There are many correct answers. Each of them refer to the same set.

- $\{3n \mid n \in \mathbb{Z}\}$,
- $\{x \mid x = 3n \text{ for some } n \in \mathbb{Z}\}$,
- $\{x \in \mathbb{Z} : x = 3n \text{ for some } n \in \mathbb{Z}\}$,
- $\{x \in \mathbb{Z} : x/3 \in \mathbb{Z}\}$.

(g) When there are many solutions for a given equation, the method of specification may be useful in the presentation of all solutions in the form of a 'solution set'. What is the set of all real solutions of the equation $\sin(x) = 0$ with unknown x?

- $\{n\pi \mid n \in \mathbb{Z}\}$,
- $\{x \in \mathbb{R} : x = n\pi \text{ for some } n \in \mathbb{Z}\}$.

Remark. This is another way to ask for the

'general solution of the equation $\sin(x) = 0$ with unknown x in $\mathbb{R}$'.

When we give the answer as

'$x = n\pi$ *where* n *is an arbitrary integer*',

what we actually mean is:

'$x = \alpha$ *is a real solution of this equation in* $\mathbb{R}$ *iff* ($\alpha = n\pi$ *for some* $n \in \mathbb{Z}$).'

(g') What is the set of all real solutions of the equation $\sin(x) = \frac{1}{2}$ with unknown x?

- $\left\{n\pi + (-1)^n \cdot \frac{\pi}{6} \;\middle|\; n \in \mathbb{Z}\right\}$,
- $\left\{x \in \mathbb{R} : x = n\pi + (-1)^n \cdot \frac{\pi}{6} \text{ for some } n \in \mathbb{Z}\right\}$.

(h) What is the set of all real solutions of the system of equation

$$(S): \begin{cases} x_1 - 5x_2 + 3x_3 = 1 \\ 2x_1 - 4x_2 + x_3 = 0 \\ x_1 + x_2 - 2x_3 = -1 \end{cases}$$

with unknown x_1, x_2, x_3 in $\mathbb{R}$?

- $\left\{ \begin{bmatrix} -2/3 + (7/6)t \\ -1/3 + (5/6)t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$,
- $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; \begin{array}{l} \text{There exists some } t \in \mathbb{R} \text{ such that} \\ x_1 = -\dfrac{2}{3} + \dfrac{7}{6}t \text{ and } x_2 = -\dfrac{1}{3} + \dfrac{5}{6}t \text{ and } x_3 = t \end{array} \right\}$.

Remark. What we are saying, without using the jargon of set language, is that '$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *is a solution of the system* (S) *iff there exists some* $t \in \mathbb{R}$ *such that* $x_1 = -\dfrac{2}{3} + \dfrac{7}{6}t$ *and* $x_2 = -\dfrac{1}{3} + \dfrac{5}{6}t$ *and* $x_3 = t$.'

(h') The method of specification is used extensively in constructions in *linear algebra*.

Below are the simplest examples:

- Let H be a $(m \times n)$-matrix with real entries.
  The null space of H is
  $$\{\mathbf{x} \in \mathbb{R}^n : H\mathbf{x} = \mathbf{0}.\}.$$
- Let H be a $(m \times n)$-matrix with real entries.
  The column space of H is
  $$\left\{ \mathbf{y} \in \mathbb{R}^m : \begin{array}{l} \text{There exists some } \mathbf{x} \in \mathbb{R}^n \\ \text{such that } \mathbf{y} = H\mathbf{x}. \end{array} \right\}.$$
- Let V be a subspace of $\mathbb{R}^n$.
  The orthogonal complement of V is
  $$\{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for any } \mathbf{x} \in V.\}.$$

Each of them can be generalized in a natural way with the help of notion of linear transformation.

(i) What is the set $\{x \mid x \neq x\}$?

This is the empty set.

Reason?

'$x \neq x$' is false no matter what x is.

Warning. We can formally construct, using the method of specification, the objects

$$\{x \mid x = x\}, \qquad\qquad \{x \mid x \notin x\}$$

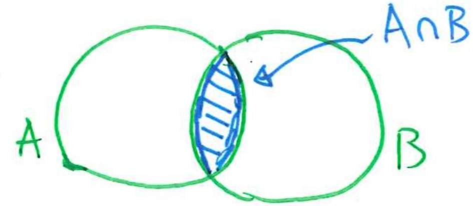
We would expect these objects to be sets. However, it will turn out that they cannot be 'reasonably regarded as sets', if we are to insist that all sets are to obey certain laws which look natural and which govern their behaviour.

6. **Definitions of the basic set operations, with the help of the Method of Specification.**

Let A, B be sets.

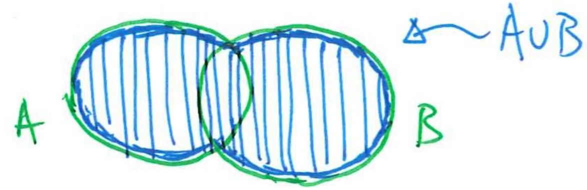
(a) The **intersection** of the sets A, B is defined to be the set

$$\{x \mid x \in A \text{ and } x \in B\}.$$



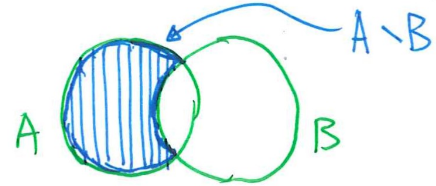
(b) The **union** of the sets A, B is defined to be the set

$$\{x \mid x \in A \text{ or } x \in B\}.$$



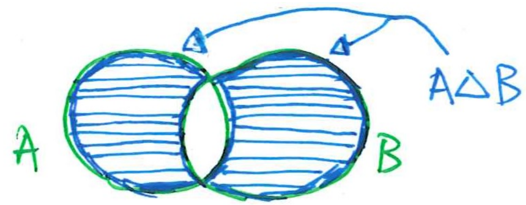
(c) The **complement of the set B in the set A** is defined to be the set

$$\{x \mid x \in A \text{ and } x \notin B\}.$$



(d) The **symmetric difference** of the sets A, B is defined to be the set

$$(A\backslash B) \cup (B\backslash A).$$



They are denoted by $A \cap B$, $A \cup B$, $A\backslash B$, $A \triangle B$ respectively.

7. **Formal definition of the notions of 'set equality', 'subset relation'.**

Recall: any two sets A, B are equal to each other as sets exactly when

'each of A, B contains as its elements every element of the other'.

Same as the above, but more clumsy:

'every element of A is an element of B and every element of B is an element of A'.

Formal definition for the notion of 'set equality':

- *Let A, B be sets. We say A is* **equal** *to B if both statements* $(\dagger), (\ddagger)$ *hold:*

  $(\dagger)$ *For any object x, [if $(x \in A)$ then $(x \in B)$].*

  $(\ddagger)$ *For any object y, [if $(y \in B)$ then $(y \in A)$].*

  We write $A = B$.

Formal and clumsy though it looks, it is best to work with this definition in calculations or proofs, because its logical content has been spelt out explicitly.

Formal definition for the notion of 'subset relation':

- *Let A, B be sets. We say A is a* **subset** *of B if the statement* $(\dagger)$ *holds:*

  $(\dagger)$ *For any object x, [if $(x \in A)$ then $(x \in B)$].*

  We write $A \subset B$ (or $B \supset A$).

8. **Definition**.

Let A be a set. The **power set** *of the set A is defined to be the set*

$$\{S \mid S \text{ is a subset of } A\}.$$

It is denoted by $\mathfrak{P}(A)$.

Remark. By definition, $S \in \mathfrak{P}(A)$ iff $S \subset A$.

9. **Example (1).**

$A = ?$	Elements of A?	Subsets of A? Elements of $\mathfrak{P}(A)$?	$\mathfrak{P}(A) = ?$
$\emptyset$	A has no element.	ϕ	$\{\phi\}$
$\{0\}$	0	$\phi, \{0\}$	$\{\phi, \{0\}\}$
$\{0,1\}$	$0, 1$	$\phi, \{0\}, \{1\}, \{0,1\}$	$\{\phi, \{0\}, \{1\}, \{0,1\}\}$
$\{0,1,2\}$	$0, 1, 2$	$\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}$	$\{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset\}$	ϕ	$\phi, \{\phi\}$	$\{\phi, \{\phi\}\}$
$\{\phi\} = \mathfrak{P}(\emptyset)$	ϕ	$\phi, \{\phi\}$	$\{\phi, \{\phi\}\}$
$\{\phi, \{\phi\}\} = \mathfrak{P}(\{\emptyset\})$	$\phi, \{\phi\}$	$\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}$	$\{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$
$\{\phi, \{\{\phi\}\}\} = \mathfrak{P}(\{\{\emptyset\}\})$	$\phi, \{\{\phi\}\}$	$\phi, \{\phi\}, \{\{\{\phi\}\}\}, \{\phi, \{\{\phi\}\}\}$	$\{\phi, \{\phi\}, \{\{\{\phi\}\}\}, \{\phi, \{\{\phi\}\}\}\}$

Remarks.

(1) $\emptyset$, $\{\emptyset\}$ are different objects.

ϕ has no element.

$\{\phi\}$ is a singleton: it contains exactly one element, namely ϕ.

(2) In general, when A has exactly N elements, $\mathfrak{P}(A)$ will have exactly

2^N elements.

Proof? Apply mathematical induction.

Example (2).

(a) What is $\mathfrak{P}(\mathfrak{P}(\emptyset))$ explicitly?

Ask: what is $\mathcal{P}(\phi)$? $\mathcal{P}(\phi) = \{\phi\}$.

Then ask: what is $\mathcal{P}(\mathcal{P}(\phi))$? $\mathcal{P}(\mathcal{P}(\phi)) = \mathcal{P}(\{\phi\})$

$= \{\phi, \{\phi\}\}$.

(b) What is $\mathfrak{P}(\mathfrak{P}(\{\emptyset\}))$ explicitly?

Ask: what is $\mathcal{P}(\{\phi\})$? $\mathcal{P}(\{\phi\}) = \{\phi, \{\phi\}\}$.

Then ask: what is $\mathcal{P}(\mathcal{P}(\{\phi\}))$? $\mathcal{P}(\mathcal{P}(\{\phi\})) = \mathcal{P}(\{\phi, \{\phi\}\})$

$= \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$.

(c) What is $\mathfrak{P}(\mathfrak{P}(\{\{\emptyset\}\}))$ explicitly?

Ask: what is $\mathcal{P}(\{\{\phi\}\})$?

Then ask: what is $\mathcal{P}(\mathcal{P}(\{\{\phi\}\}))$?

10. **Properties of the basic set operations**.

Theorem (I). *The following statements hold:*

(1) *Let A be a set.* $A \subset A$.

(2) *Let A, B be sets.* $A = B$ *iff* $[(A \subset B)$ *and* $(B \subset A)]$.

(3) *Let A, B, C be sets. Suppose* $A \subset B$ *and* $B \subset C$. *Then* $A \subset C$.

Theorem (II). *Let A, B be sets. The following statements hold:*

(1) $A \cap B \subset A$.

(2) $A \cap B \subset B$.

(3) $A \backslash B \subset A$.

(4) $A \subset A \cup B$.

(5) $B \subset A \cup B$.

Theorem (III). *Let A be a set. The following statements hold:*

(1) $\emptyset \subset A$.

(2) $A \cap \emptyset = \emptyset$.

(3) $A \cup \emptyset = A$.

(4) $A \backslash \emptyset = A$.

(5) $\emptyset \backslash A = \emptyset$.

(6) $A \triangle \emptyset = A$.

(7) $A \triangle A = \emptyset$.

Theorem (IV). *The following statements hold:*

(1) *Let* A, B, S *be sets. Suppose* $S \subset A$ *and* $S \subset B$*. Then* $S \subset A \cap B$*.*

(2) *Let* A, B, S *be sets. Suppose* $S \subset A$ *or* $S \subset B$*. Then* $S \subset A \cup B$*.*

(3) *Let* A, B, T *be sets. Suppose* $A \subset T$ *and* $B \subset T$*. Then* $A \cup B \subset T$*.*

(4) *Let* A, B, T *be sets. Suppose* $A \subset T$ *or* $B \subset T$*. Then* $A \cap B \subset T$*.*

Theorem (V). *Let* A, B, C *be sets. The following statements hold:*

(1)	$A \cap A = A.$	(1')	$A \cup A = A.$
(2)	$A \cap B = B \cap A.$	(2')	$A \cup B = B \cup A.$
(3)	$(A \cap B) \cap C = A \cap (B \cap C).$	(3')	$(A \cup B) \cup C = A \cup (B \cup C).$
(4)	$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$	(4')	$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$
(5)	$(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C).$	(5')	$(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C).$
(6)	$A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C).$	(6')	$A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C).$
(7)	$A \triangle B = (A \cup B) \backslash (A \cap B).$		
(8)	$A \triangle B = B \triangle A.$		
(9)	$(A \triangle B) \triangle C = A \triangle (B \triangle C).$		