MATH1050 Basic results on divisibility

1. Here is a (probably not exhaustive) list of properties of the integer system needed in the proofs of the results below. We have been tacitly assuming them (together with some others not mentioned here) since school-days. We will (be made to) look into them and formulate them more carefully in order to study them when we are doing an *algebra* course or an *analysis* course.

   (a) *The sum, the difference, and the product of any two (not necessarily distinct) integers are integers.*
   In symbols, this reads:

   Let $x, y \in \mathbb{Z}$. $x + y \in \mathbb{Z}$, $x - y \in \mathbb{Z}$, and $xy \in \mathbb{Z}$.

   These 'operations' obey certain 'laws of arithmetic' which we have learnt and accepted since school days.

   (b) *The sum and the product of any two (not necessarily distinct) positive integers are positive integers. Moreover, every integer is either positive or negative or zero.*

2. **Definition**.

   Let $u, v \in \mathbb{Z}$. u is said to be **divisible** *by v if there exists some $k \in \mathbb{Z}$ such that $u = kv$.*

   Remark. Before you start considering for a given pair of objects u, v whether it is true that u is divisible by v, you have to make sure that u, v are integers in the first place.

   Examples.

   - 6 is divisible by 2. Reason: $6 = 3 \cdot 2$ and 3 is an integer.
   - 8 is divisible by -2. Reason: $8 = (-4) \cdot (-2)$ and -4 is an integer.

   Further remark.

   According to definition, 0 is divisible by 0. However, no integer except 0 is divisible by 0.

3. **Theorem (1). (Properties of divisibility.)**

   The following statements hold:

   (a) *Suppose $x \in \mathbb{Z}$. Then x is divisible by x.*
   (b) *Let $x, y \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by x. Then $|x| = |y|$.*
   (c) *Let $x, y, z \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by z. Then x is divisible by z.*

   We are going to prove Statement (a) and Statement (c). The proof of Statement (b) is left as an exercise.

4. **Proof of Statement (a) of Theorem (1)**.

   Let $x \in \mathbb{Z}$. [What to prove? Un-wrap definition.]
   $x = 1 \cdot x$.
   Note that $1 \in \mathbb{Z}$.
   [So there indeed exists some $k \in \mathbb{Z}$, namely, $k = 1$, such that $x = kx$.]
   Hence x is divisible by x.

   Very formal proof of Statement (a) of Theorem (1).

   I. Let $x \in \mathbb{Z}$. [Assumption.]

   II. $x = 1 \cdot x$. [**I**, laws of arithmetic.]

   III. $1 \in \mathbb{Z}$. [Property of the number 1.]

   IV. $x = xq$ for some $q \in \mathbb{Z}$, namely $q = 1$. [**II**, **III**.]

   V. x is divisible by x. [**IV**, definition of divisibility.]

5. **Proof of Statement (c) of Theorem (1)**.

   Let $x, y, z \in \mathbb{Z}$.
   Suppose x is divisible by y and y is divisible by z. [What to deduce? What is the objective?]
   Since x is divisible by y, there exists some $g \in \mathbb{Z}$ such that $x = gy$.
   Since y is divisible by z, there exists some $h \in \mathbb{Z}$ such that $y = hz$.
   Now $x = ghz$.
   Since $g, h \in \mathbb{Z}$, we have $gh \in \mathbb{Z}$.
   [So there indeed exists some $k \in \mathbb{Z}$, namely, $k = gh$, such that $x = kz$.]
   Then x is divisible by z.

Very formal proof of Statement (c) of Theorem (1).

I. Let $x, y \in \mathbb{Z}$. [Assumption.]

II. Suppose x is divisible by y and y is divisible by z. [Assumption.]

III. x is divisible by y. [**II**.]

IV. There exists some $g \in \mathbb{Z}$ such that $x = gy$. [**III**, definition of divisibility.]

IVi. $x = gy$. [**IV**.]

IVii. $g \in \mathbb{Z}$. [**IV**.]

V. y is divisible by z. [**II**.]

VI. There exists some $h \in \mathbb{Z}$ such that $y = hz$. [**III**, definition of divisibility.]

VIi. $y = hz$. [**VI**.]

VIii. $h \in \mathbb{Z}$. [**VI**.]

VII. $x = gy$ and $y = hz$. [**IVi**, **VIi**.]

VIII. $x = ghz$. [**VII**.]

IX. $g \in \mathbb{Z}$ and $h \in \mathbb{Z}$. [**IVii**, **VIii**.]

X. $gh \in \mathbb{Z}$. [**I**, laws of arithmetic.]

XI. There exists some $k \in \mathbb{Z}$, namely, $k = gh$, such that $x = kz$. [**VIII**, **X**.]

XII. x is divisible by z. [**XI**, definition of divisibility.]

6. **Theorem (2). (Further properties of divisibility.)**

Let $n \in \mathbb{Z}$. The following statements hold:

(a) *Let $x, y \in \mathbb{Z}$. Suppose x is divisible by n and y is divisible by n. Then $x + y$ is divisible by n.*

(b) *Let $x, y \in \mathbb{Z}$. Suppose x is divisible by n or y is divisible by n. Then xy is divisible by n.*

Proof. Exercise.