1. We have been assuming these things tacitly since school days:

(a) *The sum, the difference, and the product of any two (not necessarily distinct) integers are integers.*

In symbols, this reads:

Let $x, y \in \mathbb{Z}$. $x + y \in \mathbb{Z}$, $x - y \in \mathbb{Z}$, *and* $xy \in \mathbb{Z}$.

(These 'operations' obey certain 'laws of arithmetic' which we have learnt and accepted since school days.)

(b) *The sum and the product of any two (not necessarily distinct) positive integers are positive integers.*

Moreover, every integer is either positive or negative or zero.

This contains three pieces of information:

① The already present u, v 'generates' this k, whose exact value depends on the values of u, v.

② $k \in \mathbb{Z}$.

③ u, v, k are related by $u = kv$.

2. **Definition**.

Let $u, v \in \mathbb{Z}$.

u is said to be **divisible** *by v if there exists some $k \in \mathbb{Z}$ such that $u = kv$.*

Remark. Before you start considering for a given pair of objects u, v whether it is true that u is divisible by v, you have to make sure that u, v are integers in the first place.

Examples.

- 6 is divisible by 2. Reason: $3 \in \mathbb{Z}$ and $6 = 3 \cdot 2$.
- 8 is divisible by -2. Reason: $-4 \in \mathbb{Z}$ and $8 = (-4) \cdot (-2)$

Further remark.

According to definition, 0 is divisible by 0. Reason: $1 \in \mathbb{Z}$ and $0 = 1 \cdot 0$.

However, no integer except 0 is divisible by 0.

Reason?

Claim: 'Let $u \in \mathbb{Z}$. Suppose u is divisible by 0. Then $u = 0$.'

Justification according to definition?

3. **Theorem (1). (Properties of divisibility).**

The following statements hold:

(a) *Suppose $x \in \mathbb{Z}$. Then x is divisible by x.*

(b) *Let $x, y \in \mathbb{Z}$.*
Suppose x is divisible by y and y is divisible by x.
Then $|x| = |y|$.

(c) *Let $x, y, z \in \mathbb{Z}$.*
Suppose x is divisible by y and y is divisible by z.
Then x is divisible by z.

We are going to prove Statement (a) and Statement (c). The proof of Statement (b) is left as an exercise.

4. **Proof of Statement (a) of Theorem (1).**

Let $x \in \mathbb{Z}$. [What to prove? Un-wrap definition.]

[Want to prove: 'x is divisible by x.']

$x = 1 \cdot x$

Note that $1 \in \mathbb{Z}$.

[So there exits some $k \in \mathbb{Z}$, namely $k = 1$, such that $x = k \cdot x$.]

Hence x is divisible by x. $\square$

[Roughwork.

We actually want to prove:

'there exits some $k \in \mathbb{Z}$ such that $x = k \cdot x$.'

How to reach this objective?

- Name an appropriate k which simultaneously satisfies:

$$\begin{cases} (\star) & k \in \mathbb{Z} \\ (\star\star) & x = k \cdot x. \end{cases}$$
]

Very formal proof of Statement (a) of Theorem (1).

I. Let $x \in \mathbb{Z}$. [Assumption.]

II. $x = 1 \cdot x$. [**I**, laws of arithmetic.]

III. $1 \in \mathbb{Z}$. [Property of the number 1.]

IV. $x = xq$ for some $q \in \mathbb{Z}$, namely $q = 1$. [**II**, **III**.]

V. x is divisible by x. [**IV**, definition of divisibility.]

5. **Proof of Statement (c) of Theorem (1).**

Let $x, y, z \in \mathbb{Z}$.

Suppose x is divisible by y and y is divisible by z. [What to deduce? What is the objective?]

[Want to deduce: 'x is divisible by z'.]

Since x is divisible by y,
there exists some $g \in \mathbb{Z}$ such that $x = gy$.

Since y is divisible by z,
there exists some $h \in \mathbb{Z}$ such that $y = hz$.

Now
$$x = gy = g(hz) = (gh)z.$$

Since $g \in \mathbb{Z}$ and $h \in \mathbb{Z}$,
we have $gh \in \mathbb{Z}$.

[So there exists some $k \in \mathbb{Z}$,
namely $k = gh$, such that $x = kz$.]

Then x is divisible by z. $\square$

Roughwork.

We actually want to deduce:
'there exists some $k \in \mathbb{Z}$ such that $x = kz$.'

How to reach this objective?

- Name an appropriate k which simultaneously satisfies:

$$\begin{cases} (\star) & k \in \mathbb{Z} \\ (\star\star) & x = k \cdot z. \end{cases}$$

But how to conceive such a 'k'?

- Look for candidate(s) for such a 'k' out of the information provided by the assumptions.

Further roughwork.

Now given: $g, h \in \mathbb{Z}$ and $x = gy$ and $y = hz$.

Can we relate x with z directly?

Yes: $x = gy = g(hz) = (gh)z$.

- So a good candidate for k is gh.

Very formal proof of Statement (c) of Theorem (1).

I. Let $x, y \in \mathbb{Z}$. [Assumption.]

II. Suppose x is divisible by y and y is divisible by z. [Assumption.]

III. x is divisible by y. [**II**.]

IV. There exists some $g \in \mathbb{Z}$ such that $x = gy$. [**III**, definition of divisibility.]

  IVi. $x = gy$. [**IV**.]

  IVii. $g \in \mathbb{Z}$. [**IV**.]

V. y is divisible by z. [**II**.]

VI. There exists some $h \in \mathbb{Z}$ such that $y = hz$. [**III**, definition of divisibility.]

  VIi. $y = hz$. [**VI**.]

  VIii. $h \in \mathbb{Z}$. [**VI**.]

VII. $x = gy$ and $y = hz$. [**IVi**, **VIi**.]

VIII. $x = ghz$. [**VII**.]

IX. $g \in \mathbb{Z}$ and $h \in \mathbb{Z}$. [**IVii**, **VIii**.]

X. $gh \in \mathbb{Z}$. [**I**, laws of arithmetic.]

XI. There exists some $k \in \mathbb{Z}$, namely, $k = gh$, such that $x = kz$. [**VIII**, **X**.]

XII. x is divisible by z. [**XI**, definition of divisibility.]

6. **Theorem (2) (Further properties of divisibility).**

Let $n \in \mathbb{Z}$. The following statements hold:

(a) *Let $x, y \in \mathbb{Z}$.*
Suppose x is divisible by n and y is divisible by n.
Then $x + y$ is divisible by n.

(b) *Let $x, y \in \mathbb{Z}$.*
Suppose x is divisible by n or y is divisible by n.
Then xy is divisible by n.

Proof. Exercise.