1. Here is a (probably not exhaustive) list of properties of the real number system which (together with some others not mentioned in the list) we have been tacitly assuming since school-days.

   (a) *The sum, the difference, and the product of any two (not necessarily distinct) real numbers are real numbers. The quotient of any real number by any non-zero real number is a real number is a real number. (These 'operations' obey certain 'laws of arithmetic' which we have learnt and accepted since schooldays.)*

   (b) *The sum and the product of any two (not necessarily distinct) positive real numbers are positive real numbers. If the product of two real numbers is positive, then the two real numbers are both positive or both negative. If the product of two real numbers is negative, then one of them is positive and the other is negative. The square of any non-zero real number is positive. The quotient of any one positive real number by another (not necessarily distinct) positive real number is a positive real number. Moreover, every real number is either positive or negative or zero.*

   (c) *For each positive real number x, for each integer $n \geq 2$, there exists some positive real number r such that $x = r^n$. We denote this r by $\sqrt[n]{x}$ and call it the n-th real root of x.*

   With the help of these 'rules' above, we are going to prove the inequalities below. But we will (be made to) look into these 'rules' and formulate them more carefully in order to study them (when we are doing an *analysis* course.)

2. **Statement (A1).**

   Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.

   Proof of Statement (A1).

   Let x, y be positive real numbers. Suppose $x^2 > y^2$.
   Then $x^2 - y^2 > 0$.
   Note that $x^2 - y^2 = (x - y)(x + y)$.
   Then $(x - y)(x + y) > 0$.
   Therefore $(x - y > 0$ and $x + y > 0)$ or $(x - y < 0$ and $x + y < 0)$.
   Since $x > 0$ and $y > 0$, we have $x + y > 0$.
   Then $x - y > 0$ and $x + y > 0$.
   In particular $x - y > 0$.
   Therefore $x > y$.

   Very formal proof of Statement (A1).

   I. Let x, y be positive real numbers. [Assumption.]

   II. Suppose $x^2 > y^2$. [Assumption.]

   III. $x^2 - y^2 > 0$. [**II**.]

   IV. $x^2 - y^2 = (x - y)(x + y)$. [Properties of the reals.]

   V. $(x - y)(x + y) > 0$. [**III**, **IV**.]

   VI $(x - y > 0$ and $x + y > 0)$ or $(x - y < 0$ and $x + y < 0)$. [**V**, properties of the reals.]

   VII. $x + y > 0$ [**I**.]

   VIII. $x - y > 0$. [**VI**, **VII**.]

   IX. $x > y$. [**VIII**.]

3. **Statement (A2).**

   Let x, y be positive real numbers. Suppose $x^2 \geq y^2$. Then $x \geq y$.

   Proof of Statement (A2).

   Let x, y be positive real numbers. Suppose $x^2 \geq y^2$.
   Then $x^2 - y^2 \geq 0$.
   Note that $x^2 - y^2 = (x - y)(x + y)$.
   Then $(x - y)(x + y) \geq 0$.

   Since $x > 0$ and $y > 0$, we have $x + y > 0$. Therefore $\dfrac{1}{x + y} > 0$ also.

   Then $x - y = [(x - y)(x + y)] \cdot \dfrac{1}{x + y} \geq 0$.

   Therefore $x \geq y$.

4. **Statement (B).**

Suppose x, y are positive real numbers. Then $\frac{x+y}{2} \geq \sqrt{xy}$.

Proof of Statement (B).

Suppose x, y are positive real numbers.
Then $\sqrt{x}, \sqrt{y}$ are well-defined as real numbers.
Therefore $\sqrt{x} - \sqrt{y}$ is well-defined as a real number.
Since x, y are positive, xy is positive. Then $\sqrt{xy}$ is well defined and $\sqrt{x}\sqrt{y} = \sqrt{xy}$.
Since x, y are positive, we have $(\sqrt{x})^2 = x$ and $(\sqrt{y})^2 = y$.
Therefore $x + y - 2\sqrt{xy} = (\sqrt{x})^2 - 2\sqrt{x}\sqrt{y} + (\sqrt{y})^2 = (\sqrt{x} - \sqrt{y})^2 \geq 0$.
Hence $\frac{x+y}{2} \geq \sqrt{xy}$.

Very formal proof of Statement (B).

I. Suppose x, y are positive real numbers. [Assumption.]

II. $\sqrt{x}, \sqrt{y}$ are well-defined as real numbers. [**I**.]

III. $\sqrt{x} - \sqrt{y}$ is well-defined as a real number. [**II**.]

IV. xy is a positive real number. [**I**, properties of the reals.]

V. $\sqrt{xy}$ is well-defined as a real number. [**IV**.]

VI. $\sqrt{x}\sqrt{y} = \sqrt{xy}$. [**II**, **V**, properties of the reals.]

VII. $(\sqrt{x})^2 = x$. [**I**, **II**.]

VIII. $(\sqrt{y})^2 = y$. [**I**, **II**.]

IX. $(\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y$. [**VI**, **VII**, **VIII**.]

X.$(\sqrt{x} - \sqrt{y})^2 \geq 0$. [**III**, properties of the reals.]

XI. $x - 2\sqrt{xy} + y \geq 0$. [**IX**, **X**.]

XII. $\frac{x+y}{2} \geq \sqrt{xy}$. [**XI**.]

5. **Statement (C).**

Let $x, y \in \mathbb{R}$. Suppose $x \neq 0$ or $y \neq 0$. Then $x^2 + xy + y^2 > 0$.

Proof of Statement (C).

Let $x, y \in \mathbb{R}$. Suppose $x \neq 0$ or $y \neq 0$.

(Case 1). Suppose $x \neq 0$. Then $x^2 + xy + y^2 = \frac{3x^2}{4} + \left(\frac{x}{2} + y\right)^2 > 0 + 0 = 0$.

(Case 2). Suppose $y \neq 0$. Then $x^2 + xy + y^2 = \frac{3y^2}{4} + \left(\frac{y}{2} + x\right)^2 > 0$.

Hence, in any case, $x^2 + xy + y^2 > 0$.

Very formal proof of Statement (C).

I. Let $x, y \in \mathbb{R}$. [Assumption.]

II. Suppose $x \neq 0$ or $y \neq 0$. [Assumption.]

III.

IIIi. Suppose $x \neq 0$. [One of the possibilities in **II**.]

IIIii. $x^2 + xy + y^2 = \frac{3x^2}{4} + \left(\frac{x}{2} + y\right)^2$. [Properties the reals.]

IIIiii. $\frac{3x^2}{4} > 0$. [Properties of the reals.]

IIIiv. $\left(\frac{x}{2} + y\right)^2 \geq 0$. [Properties of the reals.]

IIIv. $x^2 + xy + y^2 > 0$. [**IIIii**, **IIIiii**, **IIIiv**, properties the reals.]

IV.

IVi. Suppose $y \neq 0$. [One of the possibilities in **II**.]

IVii. $x^2 + xy + y^2 = \frac{3y^2}{4} + \left(\frac{y}{2} + x\right)^2$. [Properties of the reals.]

IViii. $\frac{3y^2}{4} > 0$. [Properties of the reals.]

IViv. $\left(\frac{y}{2}+x\right)^2 \geq 0$. [Properties of the reals.]

IVv. $x^2+xy+y^2>0$. [**IVii**, **IViii**, **IViv**, properties of the reals.]

V. $x^2+xy+y^2>0$. [**II**, **III**, **IV**.]

6. **Statement (A')**.

Let x, y be non-negative real numbers. Suppose $x^2 \geq y^2$. Then $x \geq y$.

Proof of Statement (A').

Let x, y be non-negative real numbers. Suppose $x^2 \geq y^2$.
Then $x^2 - y^2 \geq 0$.
Note that $x^2 - y^2 = (x-y)(x+y)$.
Then $(x-y)(x+y) \geq 0$.
Since $x \geq 0$ and $y \geq 0$, we have $x+y \geq 0$.
Then $x+y>0$ or $x+y=0$.

(Case 1). Suppose $x+y>0$. Since $(x-y)(x+y) \geq 0$, we have $x-y \geq 0$. Therefore $x \geq y$.
(Case 2). Suppose $x+y=0$. Since $x \geq 0$ and $y \geq 0$, we have $x=y=0$. Therefore $x \geq y$.

Hence, in any case, $x \geq y$.

Very formal proof of Statement (A').

I. Let x, y be non-negative real numbers. [Assumption.]

II. Suppose $x^2 \geq y^2$. [Assumption.]

III. $x^2 - y^2 \geq 0$. [**II**.]

IV. $x^2 - y^2 = (x-y)(x+y)$. [Properties of the reals.]

V. $(x-y)(x+y) \geq 0$. [**III**, **IV**.]

VI ($x-y \geq 0$ and $x+y \geq 0$) or ($x-y \leq 0$ and $x+y \leq 0$). [**V**, properties of the reals.]

VII. $x+y \geq 0$. [**I**.]

VIII. $x+y>0$ or $x+y=0$. [**VII**.]

IX.

IXi. Suppose $x+y>0$. [One of the possibilities in **VIII**.]
IXii. $x-y \geq 0$. [**VI**, **IXi**.]
IXiii. $x \geq y$. [**IXii**.]

X.

Xi. Suppose $x+y=0$. [One of the possibilities in **VIII**.]
Xii. $x=y=0$. [**I**, **Xi**.]
Xiii. $x \geq y$. [**Xii**.]

XI. $x \geq y$. [**VIII**, **IX**, **X**.]

7. **Statement (D). (Bernoulli's Inequality.)**

Let $m \in \mathbb{N}\backslash\{0,1\}$ and $\beta \in \mathbb{R}$. Suppose $\beta > 0$ or $-1 < \beta < 0$. Then $(1+\beta)^m > 1 + m\beta$.

Proof of Statement (D).

Let $m \in \mathbb{N}\backslash\{0,1\}$ and $\beta \in \mathbb{R}$. Suppose $\beta > 0$ or $-1 < \beta < 0$. Note that

$$\begin{aligned} (1+\beta)^m - 1 = (1+\beta)^m - 1^m &= [(1+\beta)-1][(1+\beta)^{m-1} + (1+\beta)^{m-2} + \cdots + 1] \\ &= \beta[(1+\beta)^{m-1} + (1+\beta)^{m-2} + \cdots + (1+\beta) + 1] \end{aligned}$$

(Case 1). Suppose $\beta > 0$. Then, since $\beta > 0$ and $1+\beta > 1$, we have

$$\begin{aligned} (1+\beta)^m - 1 &= \beta[(1+\beta)^{m-1} + (1+\beta)^{m-2} + \cdots + (1+\beta) + 1] \\ &> \beta\,\underbrace{(1+1+\cdots+1+1)}_{m \text{ copies}} = m\beta \end{aligned}$$

(Case 2). Suppose $-1 < \beta < 0$. Then, since $-\beta > 0$ and $0 < 1+\beta < 1$, we have

$$\begin{aligned} 1-(1+\beta)^m = -[(1+\beta)^m - 1] &= (-\beta)[(1+\beta)^{m-1} + (1+\beta)^{m-2} + \cdots + (1+\beta) + 1] \\ &< (-\beta)\,\underbrace{(1+1+\cdots+1+1)}_{m \text{ copies}} = -m\beta \end{aligned}$$

Therefore, in any cases, $(1+\beta)^m > 1 + m\beta$.

Remark. Below is a more general version of **Bernoulli's Inequality**:

Let μ be a rational number, and β be a real number. Suppose $\mu \neq 0$ and $\mu \neq 1$, and $\beta > -1$. The statements below hold:

(1) *Suppose $\mu < 0$ or $\mu > 1$. Then $(1+\beta)^\mu \geq 1 + \mu\beta$.*

(2) *Suppose $0 < \mu < 1$. Then $(1+\beta)^\mu \leq 1 + \mu\beta$.*

(3) *In each of (1), (2), equality holds iff $\beta = 0$.*

8. **Statement (E). (A 'baby version' of the Cauchy-Schwarz Inequality.)**

Suppose x, y are real numbers. Then $x^2 + y^2 \geq 2xy$. Equality holds iff $x = y$.

Proof of statement (E).

Suppose x, y are real numbers.
[Preparation. Study the difference 'L.H.S. minus R.H.S.' in the desired inequality.]
We have $(x^2+y^2) - 2xy = (x-y)^2$.

(a) Since x, y are real, $x - y$ is real.
Then $(x-y)^2 \geq 0$.
Therefore $x^2 + y^2 \geq 2xy$.

(b) i. Suppose $x = y$.
Then $(x^2+y^2) - 2xy = (x-y)^2 = (x-x)^2 = 0$.
Therefore $x^2 + y^2 = 2xy$.

ii. Suppose $x^2 + y^2 = 2xy$.
Then $0 = (x^2+y^2) - 2xy = (x-y)^2$.
Therefore $x - y = 0$. Hence $x = y$.

The result follows.

Remark. Strictly speaking, Statement (E) is not just about an inequality.

It is about a non-strict inequality together with the 'necessary and sufficient conditions for the equality to hold'.

This kind of statements is common amongst results concerned with inequalities. (For instance, see the more general version of Bernoulli's Inequality.)

9. Carefully examining the proofs of the inequalities above, we probably have to concede that we need expand the list of 'rules as regards inequalities' which we are tacitly assuming since school-days. To be more efficient, we state them with the help of symbols.

(1) *Let $x, y \in \mathbb{R}$. $y - x > 0$ iff $x < y$.*

(1*) *Let $x, y \in \mathbb{R}$. $y - x \geq 0$ iff $x \leq y$.*

(2) *Let $x, y, z \in \mathbb{R}$. If $x < y$ and $y < z$ then $x < z$.*

(2*) *Let $x, y, z \in \mathbb{R}$. The statements below hold:*

- (2*a) *$x \leq x$.*
- (2*b) *If ($x \leq y$ and $y \leq x$) then $x = y$.*
- (2*c) *If ($x \leq y$ and $y \leq z$) then $x \leq z$.*

(3) *Let $x \in \mathbb{R}$. Exactly one of '$x < 0$', '$x = 0$', '$x > 0$' is true.*

(4) *Let $x, y \in \mathbb{R}$. Suppose $x < y$. Then the statements below hold:*

- (4a) *For any $u \in \mathbb{R}$, $x + u < y + u$ and $x - u < y - u$.*
- (4b) *For any $u \in \mathbb{R}$, if $u > 0$ then $xu < yu$ and $x/u < y/u$.*
- (4c) *For any $u \in \mathbb{R}$, if $u < 0$ then $xu > yu$ and $x/u > y/u$.*

(4*) *Let $x, y \in \mathbb{R}$. Suppose $x \leq y$. Then the statements below hold:*

- (4*a) *For any $u \in \mathbb{R}$, $x + u \leq y + u$ and $x - u \leq y - u$.*
- (4*b) *For any $u \in \mathbb{R}$, if $u > 0$ then $xu \leq yu$ and $x/u \leq y/u$.*
- (4*c) *For any $u \in \mathbb{R}$, if $u < 0$ then $xu \geq yu$ and $x/u \geq y/u$.*

(5) *Let $x, y, u, v \in \mathbb{R}$. Suppose $x < y$ and $u < v$. The statements below hold:*

- (5a) *$x + u < y + v$.*
- (5b) *Further suppose $x > 0$, $y > 0$, $u > 0$ and $v > 0$. Then $xu < yv$.*

(5*) *Let $x, y, u, v \in \mathbb{R}$. Suppose $x \leq y$ and $u \leq v$.*

- (5*a) *$x + u \leq y + v$.*
- (5*b) *Further suppose $x \geq 0$, $y \geq 0$, $u \geq 0$ and $v \geq 0$. Then $xu \leq yv$.*

(6) *Let $x, y \in \mathbb{R}$. The statements below hold:*

(6a) *Suppose* $xy > 0$. *Then* ($x > 0$ *and* $y > 0$) *or* ($x < 0$ *and* $y < 0$).

(6b) *Suppose* $xy < 0$. *Then* ($x > 0$ *and* $y < 0$) *or* ($x < 0$ *and* $y > 0$).

(6*) *Let* $x, y \in \mathbb{R}$. *The statements below hold:*

(6*a) *Suppose* $xy \geq 0$. *Then* ($x \geq 0$ *and* $y \geq 0$) *or* ($x \leq 0$ *and* $y \leq 0$).

(6*b) *Suppose* $xy \leq 0$. *Then* ($x \geq 0$ *and* $y \leq 0$) *or* ($x \leq 0$ *and* $y \geq 0$).

(7) *Let* $x \in \mathbb{R}$. *Suppose* $x \neq 0$. *Then* $x^2 > 0$.

(7*) *Let* $x \in \mathbb{R}$. $x^2 \geq 0$.

We do not claim that this list is exhaustive in any sense. Nor do we claim that each item in the list is as 'basic' as each other. In fact, some of them are regarded to be more 'basic' in your *analysis* course and used in deducing others. But more fundamentally, we have side-stepped the question what we mean by the terms/phrases 'less than', 'positive'. Such questions will be resolved in your *analysis* course.