1. Tacitly assumed properties of the real number system since school-days:

(a) i. *Let* $x, y \in \mathbb{R}$. $x + y \in \mathbb{R}$ *and* $x - y \in \mathbb{R}$ *and* $xy \in \mathbb{R}$.

ii. *Let* $x, y \in \mathbb{R}$. *Suppose* $y \neq 0$. *Then* $x/y \in \mathbb{R}$.

(b) i. *Let* $x \in \mathbb{R}$. *Exactly one of* '$x < 0$', '$x = 0$', '$x > 0$' *is true.*

ii. *Let* $x, y \in \mathbb{R}$. *Suppose* $x > 0$ *and* $y > 0$. *Then* $x + y > 0$ *and* $xy > 0$ *and* $x/y > 0$.

iii. *Let* $x, y \in \mathbb{R}$. *Suppose* $xy > 0$. *Then* ($x > 0$ *and* $y > 0$) *or* ($x < 0$ *and* $y < 0$).

(c) *For each positive real number* x, *for each integer* $n \geq 2$, *there exists some positive real number* r *such that* $x = r^n$.
We denote this r *by* $\sqrt[n]{x}$ *and call it the* n*-th real root of* x.

2. **Statement (A1).**

Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.

Proof of Statement (A1). [Ask: Assumptions in the statement? Conclusions?]

[Write down the assumptions.]

Let x, y be positive real numbers.

Suppose $x^2 > y^2$.

Then $x^2 - y^2 > 0$.

Note that $x^2 - y^2 = (x-y)(x+y)$.

Then $(x-y)(x+y) > 0$.

Therefore

[Don't panic: which part of the assumptions is yet to be used?]

$(x-y>0 \text{ and } x+y>0)$ or $(x-y<0 \text{ and } x+y<0)$.

Since $x>0$ and $y>0$, we have $x+y>0$.

Then $x-y>0$ and $x+y>0$.

In particular $x-y>0$.

Therefore $x>y$. □

Roughwork.

[Ask: What do we want to deduce?

Answer: '$x>y$'.

Ask: Any equivalent formulation which may be easier to manipulate and which may seem to link with the assumptions?

Answer: '$x-y>0$'.

We observe:

① We can turn the assumption '$x^2>y^2$' into '$x^2-y^2>0$'.

② $\underbrace{x^2-y^2}_{\text{positive by assumption.}} = \underbrace{(x-y)}_{???}\underbrace{(x+y)}_{\text{positive???}}$.]

Statement (A1).

Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.

Very formal proof of Statement (A1).

I. Let x, y be positive real numbers. [Assumption.]

II. Suppose $x^2 > y^2$. [Assumption.]

III. $x^2 - y^2 > 0$. [**II**.]

IV. $x^2 - y^2 = (x - y)(x + y)$. [Properties of the reals.]

V. $(x - y)(x + y) > 0$. [**III**, **IV**.]

VI $(x - y > 0$ and $x + y > 0)$ or $(x - y < 0$ and $x + y < 0)$. [**V**, properties of the reals.]

VII. $x + y > 0$ [**I**.]

VIII. $x - y > 0$. [**VI**, **VII**.]

IX. $x > y$. [**VIII**.]

3. **Statement (A2).**

Let x, y be positive real numbers. Suppose $x^2 \geq y^2$. Then $x \geq y$.

Proof of Statement (A2).

Let x, y be positive real numbers. Suppose $x^2 \geq y^2$.

Then $x^2 - y^2 \geq 0$.

Note that $x^2 - y^2 = (x - y)(x + y)$.

Then $(x - y)(x + y) \geq 0$.

Since $x > 0$ and $y > 0$, we have $x + y > 0$. Therefore $\dfrac{1}{x+y} > 0$ also.

Then $x - y = [(x - y)(x + y)] \cdot \dfrac{1}{x+y} \geq 0$.

Therefore $x \geq y$.

4. **Statement (B).**

Suppose x, y are positive real numbers. Then $\dfrac{x+y}{2} \geq \sqrt{xy}$.

Proof of Statement (B). [Assumptions? Conclusion?]

[Assumption] → Suppose x, y are positive real numbers.

Here we make sure that everything which appears in the calculation below makes sense.

→ Then $\sqrt{x}, \sqrt{y}$ are well-defined as real numbers.

→ Also, $\sqrt{x}-\sqrt{y}$ is well-defined as a real number.

→ Since x, y are positive, xy is also positive.

→ Moreover, $\sqrt{xy}$ is well-defined as a real number, and $\sqrt{xy} = \sqrt{x}\cdot\sqrt{y}$.

→ Since x, y are positive, $x = (\sqrt{x})^2$ and $y = (\sqrt{y})^2$.

Therefore

$$\begin{aligned} x+y-2\sqrt{xy} &= (\sqrt{x})^2 + (\sqrt{y})^2 - 2\sqrt{x}\cdot\sqrt{y} \\ &= (\sqrt{x}-\sqrt{y})^2 \\ &\geq 0. \end{aligned}$$

Hence $\dfrac{x+y}{2} \geq \sqrt{xy}$. □

Ask: How to reach '$\dfrac{x+y}{2} \geq \sqrt{xy}$'?

Clueless? So ask:

Is there some equivalent formulation of

'$\dfrac{x+y}{2} \geq \sqrt{xy}$'

which is more suggestive, linking what we know or have learnt?

Or is there some consequence of '$\dfrac{x+y}{2} \geq \sqrt{xy}$' which is more suggestive?

Ask: Assuming $\dfrac{x+y}{2} \geq \sqrt{xy}$ holds, what happens?

Answer. $\dfrac{x+y}{2} \geq \sqrt{xy}$.

Then $x - 2\sqrt{xy} + y \geq 0$.

(Allowed?) → $(\sqrt{x})^2 - 2\sqrt{x}\cdot\sqrt{y} + (\sqrt{y})^2 \geq 0$.

$(\sqrt{x}-\sqrt{y})^2 \geq 0$. ← Suggestive?

Now ask: Can this process be 'reversed'?

Statement (B).

Suppose x, y are positive real numbers. Then $\dfrac{x+y}{2} \geq \sqrt{xy}$.

Very formal proof of Statement (B).

I. Suppose x, y are positive real numbers. [Assumption.]

II. $\sqrt{x}$, $\sqrt{y}$ are well-defined as real numbers. [**I**.]

III. $\sqrt{x} - \sqrt{y}$ is well-defined as a real number. [**II**.]

IV. xy is a positive real number. [**I**, properties of the reals.]

V. $\sqrt{xy}$ is well-defined as a real number. [**IV**.]

VI. $\sqrt{x}\sqrt{y} = \sqrt{xy}$. [**II**, **V**, properties of the reals.]

VII. $(\sqrt{x})^2 = x$. [**I**, **II**.]

VIII. $(\sqrt{y})^2 = y$. [**I**, **II**.]

IX. $(\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y$. [**VI**, **VII**, **VIII**.]

X.$(\sqrt{x} - \sqrt{y})^2 \geq 0$. [**III**, properties of the reals.]

XI. $x - 2\sqrt{xy} + y \geq 0$. [**IX**, **X**.]

XII. $\dfrac{x+y}{2} \geq \sqrt{xy}$. [**XI**.]

5. **Statement (C).**

Let $x, y \in \mathbb{R}$. Suppose $x \neq 0$ or $y \neq 0$. Then $x^2 + xy + y^2 > 0$.

Proof of Statement (C).

Let $x, y \in \mathbb{R}$. Suppose $x \neq 0$ or $y \neq 0$.

(Case 1). Suppose $x \neq 0$.

Then $x^2 + xy + y^2 = \frac{3x^2}{4} + \left(\frac{x}{2} + y\right)^2$

$> 0 + 0$

$= 0$.

(Case 2). Suppose $y \neq 0$.

Then $x^2 + xy + y^2 = \frac{3y^2}{4} + \left(\frac{y}{2} + x\right)^2$

$> 0 + 0$

$= 0$.

Hence, in any case, $x^2 + xy + y^2 > 0$. $\square$

Smart argument.

Note that $x^2 + xy + y^2 = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}(x+y)^2$. So ???

Ask: How to read '$x^2 + xy + y^2 > 0$' from '$x \neq 0$'?

Answer: Observe that $x^2 + xy + y^2$ is a quadratic expression.

This suggests something we have learnt: Completing the square.

Ask: Is the equality below possible?

$$x^2 + xy + y^2 = \#_1 \cdot x^2 + \#_2 \cdot \underbrace{(\cdots\cdots)}^2$$

non-negative numbers?

absorbing everything involving y?

Answer: Yes:

$$x^2 + xy + y^2 = \frac{3}{4}x^2 + 1 \cdot \left(\frac{x}{2} + y\right)^2$$

And this is positive because $x \neq 0$.

Reminder: What if $x = 0$?

6. **Statement (A').**

Let x, y be non-negative real numbers. Suppose $x^2 \geq y^2$. Then $x \geq y$.

Proof of Statement (A').

Let x, y be non-negative real numbers.

Suppose $x^2 \geq y^2$.

Then $x^2 - y^2 \geq 0$.

Note that $x^2 - y^2 = (x-y)(x+y)$.

Then $(x-y)(x+y) \geq 0$.

Therefore

$(x-y \geq 0 \text{ and } x+y \geq 0)$ or $(x-y \leq 0 \text{ and } x+y \leq 0)$.

[why? So we realize something we have tacitly assumed in school maths.]

[It is tempting to immediately conclude from '$x+y \geq 0$' that '$x-y \geq 0$'. But there is a problem. ...]

Since $x \geq 0$ and $y \geq 0$, we have $x+y \geq 0$.

Then $x+y > 0$ or $x+y = 0$.

(Case 1). Suppose $x+y > 0$.

Since $(x-y)(x+y) \geq 0$, we have $x-y \geq 0$. Then $x \geq y$.

(Case 2). Suppose $x+y = 0$.

Since $x \geq 0$ and $y \geq 0$, we have $x = y = 0$. Then $x \geq y$.

Therefore, in any case, we have $x \geq y$. □

Very formal proof of Statement (A').

I. Let x, y be non-negative real numbers. [Assumption.]

II. Suppose $x^2 \geq y^2$. [Assumption.]

III. $x^2 - y^2 \geq 0$. [**II**.]

IV. $x^2 - y^2 = (x-y)(x+y)$. [Properties of the reals.]

V. $(x-y)(x+y) \geq 0$. [**III**, **IV**.]

VI $(x - y \geq 0$ and $x + y \geq 0)$ or $(x - y \leq 0$ and $x + y \leq 0)$. [**V**, properties of the reals.]

VII. $x + y \geq 0$. [**I**.]

VIII. $x+y > 0$ or $x+y = 0$. [**VII**.]

IX.

- **IXi**. Suppose $x + y > 0$. [One of the possibilities in **VIII**.]
- **IXii**. $x - y \geq 0$. [**VI**, **IXi**.]
- **IXiii**. $x \geq y$. [**IXii**.]

X.

- **Xi**. Suppose $x + y = 0$. [One of the possibilities in **VIII**.]
- **Xii**. $x = y = 0$. [**I**, **Xi**.]
- **Xiii**. $x \geq y$. [**Xii**.]

XI. $x \geq y$. [**VIII**, **IX**, **X**.]

7. **Statement (D). (Bernoulli's Inequality.)**

Let $m \in \mathbb{N}\backslash\{0,1\}$ and $\beta \in \mathbb{R}$. Suppose $\beta > 0$ or $-1 < \beta < 0$. Then $(1+\beta)^m > 1 + m\beta$.

Proof of Statement (D).

Let $m \in \mathbb{N}\backslash\{0,1\}$ and $\beta \in \mathbb{R}$.

Suppose $\beta > 0$ (or) $-1 < \beta < 0$.

[Want to deduce : $(1+\beta)^m > 1+m\beta$.]

Roughwork.

Ask: How to arrive at '$(1+\beta)^m > 1+m\beta$'?

Any equivalent formulation which links up with what we learnt?

Answer: '$(1+\beta)^m - 1 > m\beta$'.

Recall from school maths:

$$s^n - t^n = (s-t)(s^{n-1} + s^{n-2}t + s^{n-3}t^2 + \ldots + s^2t^{n-3} + st^{n-2} + t^{n-1})$$

So?

Note that

$$\begin{aligned}(1+\beta)^m - 1 &= (1+\beta)^m - 1^m \\ &= [(1+\beta)-1][(1+\beta)^{m-1} + (1+\beta)^{m-2} + \ldots + (1+\beta) + 1] \\ &= \beta \cdot [(1+\beta)^{m-1} + (1+\beta)^{m-2} + \ldots + (1+\beta) + 1].\end{aligned}$$

(Case 1). Suppose $\beta > 0$. Then, since $\beta > 0$ and $1+\beta > 1$,

$$(1+\beta)^m - 1 = \beta \cdot \underbrace{[(1+\beta)^{m-1} + (1+\beta)^{m-2} + \ldots + (1+\beta) + 1]}_{m \text{ terms}} > \beta \cdot \underbrace{(1+1+\ldots+1+1)}_{m \text{ copies}} = m\beta.$$

(Case 2). Suppose $-1 < \beta < 0$. Then, since $-\beta > 0$ and $0 < 1+\beta < 1$,

$$1 - (1+\beta)^m = (-\beta) \cdot [(1+\beta)^{m-1} + (1+\beta)^{m-2} + \ldots + (1+\beta) + 1] < (-\beta) \cdot \underbrace{(1+1+\ldots+1+1)}_{m \text{ copies}} = -m\beta.$$

Therefore, in any case, $(1+\beta)^m > 1 + m\beta$. $\square$

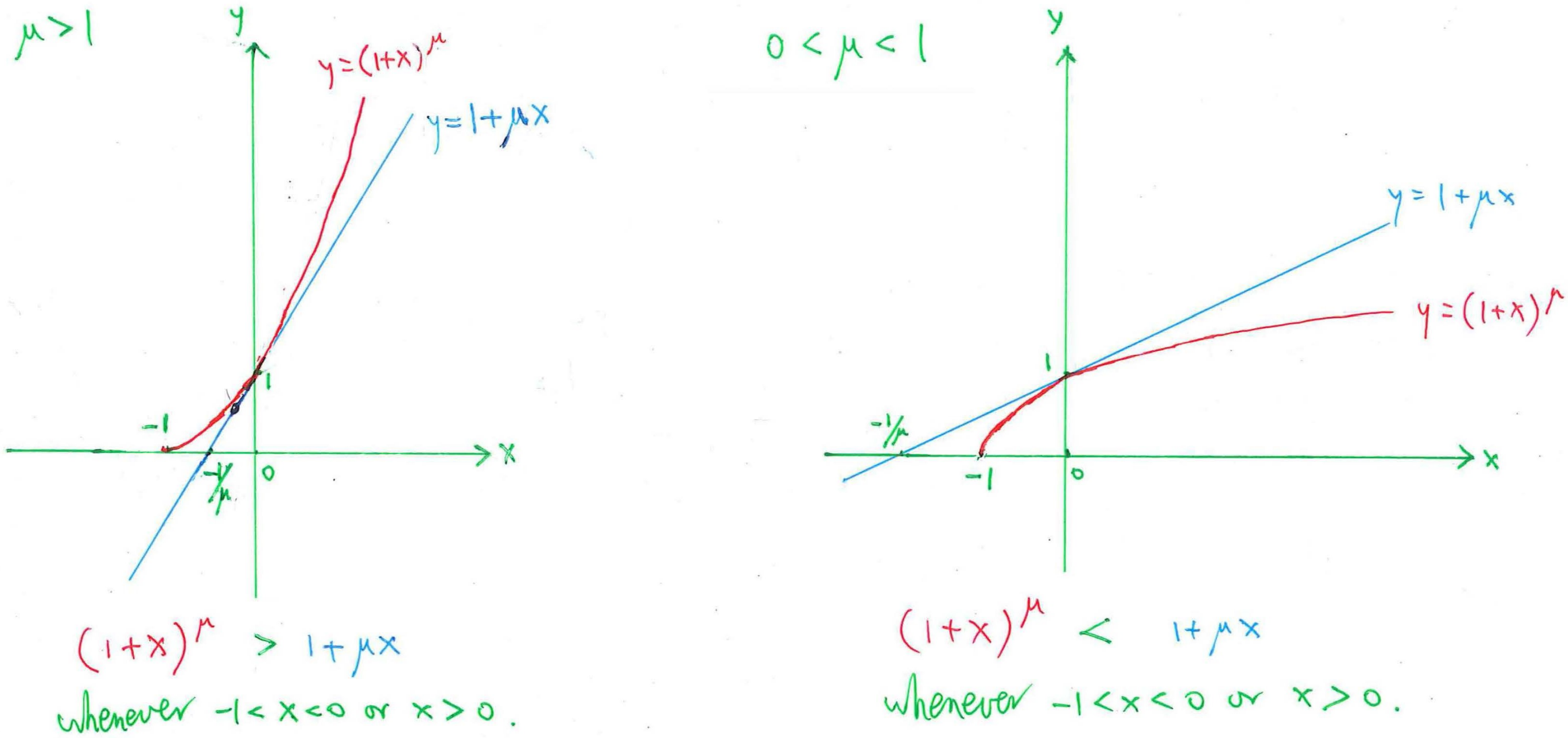
Remark. Below is a more general version of **Bernoulli's Inequality**:

Let μ be a rational number, and β be a real number.
Suppose $\mu \neq 0$ and $\mu \neq 1$, and $\beta > -1$.
The statements below hold:

(1) *Suppose $\mu < 0$ or $\mu > 1$. Then $(1+\beta)^\mu \geq 1 + \mu\beta$.*

(2) *Suppose $0 < \mu < 1$. Then $(1+\beta)^\mu \leq 1 + \mu\beta$.*

(3) *In each of (1), (2), equality holds iff $\beta = 0$.*



8. **Statement (E). (A 'baby version' of the Cauchy-Schwarz Inequality.)**

Suppose x, y are real numbers. Then $x^2 + y^2 \geq 2xy$. Equality holds iff $x = y$.

Proof of statement (E).

Suppose x, y are real numbers.

[Preparation. Study the difference 'L.H.S. minus R.H.S.' in the desired inequality.]

We have $(x^2 + y^2) - 2xy = (x - y)^2$.

(a) Since x, y are real, $x - y$ is real.
Then $(x - y)^2 \geq 0$. Therefore $x^2 + y^2 \geq 2xy$.

(b) i. Suppose $x = y$.
Then $(x^2 + y^2) - 2xy = (x - y)^2 = (x - x)^2 = 0$. Therefore $x^2 + y^2 = 2xy$.

ii. Suppose $x^2 + y^2 = 2xy$.
Then $0 = (x^2 + y^2) - 2xy = (x - y)^2$. Therefore $x - y = 0$. Hence $x = y$.

The result follows.

Remark. Strictly speaking, Statement (E) is not just about an inequality.

It is about a non-strict inequality together with the 'necessary and sufficient conditions for the equality to hold'.

This kind of statements is common amongst results concerned with inequalities. (For instance, see the more general version of Bernoulli's Inequality.)

9. We need expand the list of 'rules as regards inequalities' which we are tacitly assuming since school-days!

(1) *Let* $x, y \in \mathbb{R}$. $y - x > 0$ *iff* $x < y$.

(1*) *Let* $x, y \in \mathbb{R}$. $y - x \geq 0$ *iff* $x \leq y$.

(2) *Let* $x, y, z \in \mathbb{R}$. *If* $x < y$ *and* $y < z$ *then* $x < z$.

(2*) *Let* $x, y, z \in \mathbb{R}$. *The statements below hold:*

(2*a) $x \leq x$.

(2*b) *If* ($x \leq y$ and $y \leq x$) *then* $x = y$.

(2*c) *If* ($x \leq y$ and $y \leq z$) *then* $x \leq z$.

(3) *Let* $x \in \mathbb{R}$. *Exactly one of* '$x < 0$', '$x = 0$', '$x > 0$' *is true.*

(4) *Let* $x, y \in \mathbb{R}$. *Suppose* $x < y$. *Then the statements below hold:*

(4a) *For any* $u \in \mathbb{R}$, $x + u < y + u$ *and* $x - u < y - u$.

(4b) *For any* $u \in \mathbb{R}$, *if* $u > 0$ *then* $xu < yu$ *and* $x/u < y/u$.

(4c) *For any* $u \in \mathbb{R}$, *if* $u < 0$ *then* $xu > yu$ *and* $x/u > y/u$.

(4*) *Let* $x, y \in \mathbb{R}$. *Suppose* $x \leq y$. *Then the statements below hold:*

(4*a) *For any* $u \in \mathbb{R}$, $x + u \leq y + u$ *and* $x - u \leq y - u$.

(4*b) *For any* $u \in \mathbb{R}$, *if* $u > 0$ *then* $xu \leq yu$ *and* $x/u \leq y/u$.

(4*c) *For any* $u \in \mathbb{R}$, *if* $u < 0$ *then* $xu \geq yu$ *and* $x/u \geq y/u$.

... More rules:

(5) *Let* $x, y, u, v \in \mathbb{R}$. *Suppose* $x < y$ *and* $u < v$. *The statements below hold:*

(5a) $x + u < y + v$.

(5b) *Further suppose* $x > 0$, $y > 0$, $u > 0$ *and* $v > 0$. *Then* $xu < yv$.

(5*) *Let* $x, y, u, v \in \mathbb{R}$. *Suppose* $x \leq y$ *and* $u \leq v$.

(5$^*a$) $x + u \leq y + v$.

(5$^*b$) *Further suppose* $x \geq 0$, $y \geq 0$, $u \geq 0$ *and* $v \geq 0$. *Then* $xu \leq yv$.

(6) *Let* $x, y \in \mathbb{R}$. *The statements below hold:*

(6a) *Suppose* $xy > 0$. *Then* ($x > 0$ *and* $y > 0$) *or* ($x < 0$ *and* $y < 0$).

(6b) *Suppose* $xy < 0$. *Then* ($x > 0$ *and* $y < 0$) *or* ($x < 0$ *and* $y > 0$).

(6*) *Let* $x, y \in \mathbb{R}$. *The statements below hold:*

(6$^*a$) *Suppose* $xy \geq 0$. *Then* ($x \geq 0$ *and* $y \geq 0$) *or* ($x \leq 0$ *and* $y \leq 0$).

(6$^*b$) *Suppose* $xy \leq 0$. *Then* ($x \geq 0$ *and* $y \leq 0$) *or* ($x \leq 0$ *and* $y \geq 0$).

(7) *Let* $x \in \mathbb{R}$. *Suppose* $x \neq 0$. *Then* $x^2 > 0$.

(7*) *Let* $x \in \mathbb{R}$. $x^2 \geq 0$.