0. *You are supposed to have a lot of practical experience in* **solving equations and inequalities (and systems of such things)** *in school mathematics, and in a beginning course in linear algebra.*

   Recall:

   > A **predicate with variables** $x, y, z, \cdots$ is a statement 'modulo' the ambiguity of possibly one or several variables $x, y, z, \cdots$. In general, it may fail to be a statement. However, provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

1. **Equations and inequalities as predicates.**

   (a) Every **equation** with one unknown (or more) is a predicate in which the variables are the unknowns of the equation.
   Examples.

   i. $x^2 - 1 = 0$.
   ii. $x^2 + 1 = 0$.
   iii. $x + 2y + 3 = 0$.
   iv. $x^2 + y^2 = 1$.
   v. $x^2 - y^2 = 1$.
   vi. $x + y + z = 1$.
   vii. $x^2 + y^2 + z^2 = 1$.
   viii. $x^2 + y^2 = z^2$.

   (b) Every **inequality** with one unknown (or more) is a predicate in which the variables are the unknowns of the inequality.
   Examples.

   i. $x^2 - 1 \geq 0$.
   ii. $x^2 + 1 > 0$.
   iii. $x + 2y + 3 \leq 0$.
   iv. $x^2 + y^2 < 1$.
   v. $x^2 - y^2 < 1$.
   vi. $x + y + z \geq 1$.
   vii. $x^2 + y^2 + z^2 \leq 1$.
   viii. $x^2 + y^2 \leq z^2$.

2. **Systems of equations/inequalities.**

   A **collection of equations/inequalities** with one unknown (or more) joint by the word '*and*' and/or '*or*' is referred to as a system of equations/inequalities. Such a system may be regarded as a predicate in which the variables are the unknowns in the equations and the inequalities.

   (a) When both '*and*' and '*or*' are involved in a system, we use brackets appropriately to indicate how the system is supposed to be understood.
   Examples.

   i. $x < 3$ and $x > -1$
   ii. $x > 3$ or $x < -1$
   iii. $x + y = 1$ and $x^2 + y^2 \leq 9$
   iv. $x + y < 2$ and $x^2 + y^2 \leq 4$
   v. $x + y + z < 1$ and $x > 0$ and $y > 0$ and $z > 0$
   vi. $x + y < 3$ or $2x - y > 4$
   vii. $x > 1$ or $y < 2$ or $z > 3$
   viii. $(x < 3y$ and $y < 2z)$ or $x + y + z = 1$
   ix. $x < 3y$ and $(y < 2z$ or $x + y + z = 1)$

   (b) When only the word '*and*' is involved, we usually present the system by listing the equations/inequalities 'line-by-line', with the 'left curly bracket' placed on the left side of this list.
   Examples.

   i. $\left\{\begin{array}{rcrcl} x & + & 2y & = & -3 \\ 3x & + & y & = & 1 \end{array}\right.$

   ii. $\left\{\begin{array}{rcrcl} x & + & 2y & = & 2 \\ x^2 & + & y^2 & = & 1 \end{array}\right.$

   iii. $\left\{\begin{array}{rcrcl} x^2 & + & y^2 & = & 1 \\ x^2 & - & y^2 & = & 1 \end{array}\right.$

   iv. $\left\{\begin{array}{rcrcrcl} x & + & 2y & - & z & = & -3 \\ 3x & + & y & + & 3z & = & 1 \end{array}\right.$

   v. $\left\{\begin{array}{rcrcrcl} x^2 & + & y^2 & + & z^2 & = & 4 \\ x^2 & + & y^2 & - & z^2 & = & 1. \end{array}\right.$

   vi. $\left\{\begin{array}{rcrcrcl} x & + & y & - & z & = & 0 \\ 3x & + & y & + & 3z & = & 1 \\ 2x & - & 4y & + & z & = & 3 \end{array}\right.$

3. **What is 'solving an equation/inequality'?**

   To **solve** an equation/inequality with unknowns $x, y, z, \cdots$ amongst *so-and-so* is to specify, for that equation/inequality regarded as a predicate with variables $x, y, z, \cdots$, all the 'concrete objects' amongst *so-and-so* which, upon 'substitution into the variables' of the predicate, turn the predicate into a true statement. Each such 'concrete object' which turn the predicate into a true statement is called a **solution** for that equation/inequality.

   In practice, this is what we usually do:

   - First perform some manipulation, starting from the equation/inequality concerned, in order to find all possible candidates for $x, y, z, \cdots$.
   - Then substitute these candidates for $x, y, z, \cdots$ into the predicate (which is the equation/inequality concerned) to see whether we obtain a true statement.

4. **What is 'solving an equation/inequality with one or more unknowns in so-and-so'?**

   To solve an equation/inequality with unknowns $x, y, z, \cdots$ in *so-and-so* is to specify, for that equation/inequality regarded as a predicate with variables $x, y, z, \cdots$, all the 'concrete objects' amongst *so-and-so* which, upon 'substitution into the variables' of the predicate, turn the predicate into a true statement.

   Each such 'concrete object' which turn the predicate into a true statement is called a solution in *so-and-so* for that equation/inequality.

   Remark. In this course, the '*so-and-so*' concerned are often the reals or the complex numbers. But the same idea applies when the '*so-and-so*' concerned are the rationals, the integers, or perhaps other 'exotic objects' which you did not encounter at school.

5. **What is 'solving a system of equations/inequalities with one or more unknowns in so-and-so'?**

   To solve a system of equations/inequalities with unknowns $x, y, z, \cdots$ in *so-and-so* is to specify, for all the equations/inequalities in the system regarded as predicates with variables $x, y, z, \cdots$, the common 'concrete objects' amongst *so-and-so* which, upon 'substitution into the variables' of the predicates, turn all predicates simultaneously into true statements.

   Each such 'concrete object' which turn all predicates into a true statement is called a solution in *so-and-so* for that system of equations/inequalities.

6. **What is the solution set of an equation/inequality or a system of equations/inequalities with one or more unknowns in so-and-so?**

   The solution set of an equation/inequalities (or a system of equations/inequalities) with one unknown in *so-and-so* is the collection of all objects amongst *so-and-so* which are solutions of the equation/inequalities (or the system of equations/inequalitites) concerned.

   The solution set of an equation/inequality (or a system of equations/inequalities) with two unknowns in *so-and-so* is the collection of 'ordered pairs' of concrete objects amongst *so-and-so* which are solutions of the equation/inequalities (or the system of equations/inequalities) concerned.

   The solution set of an equation/inequality (or a system of equations/inequalities) with three unknowns in *so-and-so* is the collection of 'ordered triples' of concrete objects amongst *so-and-so* which are solutions of the equation/inequality (or the system of equations/inequalities) concerned.

   Et cetera.

7. **Illustrations: what are we really doing when we solve an equation with one unknown in the reals?**

   Here we look back from a more advanced standpoint at '**solving equations**' in school mathematics.

   (a) Below is a typical example for the topic *quadratic equations* in a typical school mathematics textbook:

   '*Solve the equation $x^2 - 3x + 2 = 0$ with unknown x in the reals.*'

   It is likely that you find such a 'chain' of calculations from the textbook:

$$\begin{array}{rcl} x^2 - 3x + 2 & = & 0 \\ (x-1)(x-2) & = & 0 \\ x - 1 = 0 & \text{or} & x - 2 = 0 \\ x = 1 & \text{or} & x = 2 \end{array}$$

This 'chain' of calculations is likely followed by a paragraph to the effect below:

We check whether '$x = 1$', '$x = 2$' are solutions of the equation $x^2 - 3x + 2 = 0$.

Put $x = 1$ into $x^2 - 3x + 2 = 0$. LHS = RHS.
Put $x = 2$ into $x^2 - 3x + 2 = 0$. LHS = RHS.

Hence we conclude that the solution of the equation $x^2 - 3x + 2 = 0$ is given by $x = 1$ or $x = 2$.

(b) Below is a typical example for the topic *absolute value* in a typical school mathematics textbook:

'*Solve the equation $3x = |2x - 3|$ with unknown x in the reals.*'

It is likely that you find such a 'chain' of calculations from the textbook:

$$\begin{array}{rcl} 3x & = & |2x-3| \\ 3x = 2x - 3 & \textit{or} & 3x = -(2x-3) \\ x = -3 & \textit{or} & x = \dfrac{3}{5} \end{array}$$

This 'chain' of calculations is definitely followed by a paragraph to the effect below: *We check whether '$x = -3$', '$x = \frac{3}{5}$' are solutions of the equation $3x = |2x - 3|$.*

Put $x = -3$ into $3x = |2x - 3|$. LHS $\neq$ RHS.

Put $x = \frac{3}{5}$ into $3x = |2x - 3|$. LHS = RHS.

Hence we conclude that the (only) solution of the equation $3x = |2x - 3|$ is given by $x = \frac{3}{5}$.

When we think very carefully on the 'logic' in the examples above, we will realize we are giving a very much garbled version of the chain of reasoning below, in the respective examples:

(a) ($*_1$) We proceed to solve for all real solutions of the equation $x^2 - 3x + 2 = 0$ —— ($\star$) with unknown x below.

($*_2$) Let α be a real number. Suppose $x = \alpha$ is a solution of ($\star$). Then:

$$\begin{array}{rcl} \alpha^2 - 3\alpha + 2 & = & 0 \\ (\alpha-1)(\alpha-2) & = & 0 \\ \alpha - 1 = 0 & \text{or} & \alpha - 2 = 0 \end{array}$$

Therefore $\alpha = 1$ or $\alpha = 2$.
(We have only argued for the statement

'*if $x = \alpha$ is a solution of ($\star$) then $\alpha = 1$ or $\alpha = 2$.*'

We cannot immediately conclude that $x = 1$ is a solution of ($\star$); nor can we conclude that $x = 2$ is a solution of the equation ($\star$).)

($*_3$) (Now we check whether $x = 1$ is a solution of ($\star$). Then we check whether $x = 2$ is a solution of ($\star$).)

- Suppose $\alpha = 1$. Then $\alpha^2 - 3\alpha + 2 = 1^2 - 3(1) + 2 = 0$.
  It follows that $x = 1$ is a solution of the equation ($\star$).
- Suppose $\alpha = 2$. Then $\alpha^2 - 3\alpha + 2 = 2^2 - 3(2) + 2 = 0$.
  It follows that $x = 2$ is a solution of the equation ($\star$).

($*_4$) It follows (from ($*_2$), ($*_3$) combined) that the solution of ($\star$) is $x = 1$ or $x = 2$.

(b) ($*_1$) We proceed to solve for all real solutions of the equation $3x = |2x - 3|$ —— ($\star$) with unknown x below.

($*_2$) Let α be a real number. Suppose $x = \alpha$ is a solution of ($\star$). Then:

$$\begin{array}{rcl} 3\alpha & = & |2\alpha - 3| \\ 3\alpha = 2\alpha - 3 & \text{or} & 3\alpha = -(2\alpha - 3) \end{array}$$

Therefore $\alpha = -3$ or $\alpha = \frac{3}{5}$.
(We have only argued for the statement

'*if* $x = \alpha$ *is a solution of* $(\star)$ *then* $\alpha = -3$ *or* $\alpha = \frac{3}{5}$.'

We cannot immediately conclude that $x = -3$ is a solution of $(\star)$; nor can we conclude that $x = \frac{3}{5}$ is a solution of the equation $(\star)$.)

$(*_3)$ (Now we check whether $x = -3$ is a solution of $(\star)$. Then we check whether $x = \frac{3}{5}$ is a solution of $(\star)$.)

- Suppose $\alpha = -3$. Then $3\alpha = 3(-3) = -9 \neq 9 = |2 \cdot (-3) - 3| = |2\alpha - 3|$.
  It follows that $x = -3$ is not a solution of the equation $(\star)$.
- Suppose $\alpha = \frac{3}{5}$. Then $3\alpha = 3 \cdot \frac{3}{5} = \frac{9}{5} = |2 \cdot \frac{3}{5} - 3| = |2\alpha - 3|$.
  It follows that $x = \frac{3}{5}$ is a solution of the equation $(\star)$.

$(*_4)$ It follows (from $(*_2), (*_3)$ combined) that the only solution of $(\star)$ is $x = \frac{3}{5}$.

What we are actually seeing in these school maths examples is a schema for solving an equation with one unknown in the reals:

- (Step 1.) First perform some manipulation, starting from the equation concerned, in order to find all possible candidates for the unknown x in the reals.

  In terms of logic, this is what we are telling others after the completion of this step:

  Let α *be a real number. Suppose '*$x = \alpha$*' is a solution of this equation concerned. Then the real number* α *is of value this-or-that-or-blah-blah-blah.'*

  Be careful:

  * If it happens that there is no candidate amongst the reals to talk about, we declare that there is no real solution for the equation concerned.
  * It can happen that some false candidates are found because of the nature of certain kinds of manipulations (for example, '*squaring both sides of the equation*', '*multiplying both sides of the equation by an expression involving the unknowns*', '*removing logarithm by exponentiating both sides of the equations*'.) Such false candidates need be removed by 'checking solution' (which is Step 2 below).

- (Step 2.) Substitute each candidate for x named above (in Step 1) into the predicate (which is the equation concerned) to see whether we obtain a true statement.

  * If we obtain a true statement from such a candidate, we declare it to be a solution of the equation concerned.
  * If we obtain a false statement from such a candidate, we declare it to be not a solution of the equation concerned.

  In terms of logic, this is what we are telling others after the completion of this step:

  Let α *be a real number. Suppose* α *is of value this-or-that-or-blah-blah-blah. Then '*$x = \alpha$*' is a solution of the equation concerned.'*

Remark. The process for solving a system of equations/inequalities with one or more unknowns in *so-and-so* is analogous to what has been described.