1. **Definition. (Inner product in $\mathbb{R}^n$.)**

For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *write* $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t \mathbf{v}$.

$\langle \mathbf{u}, \mathbf{v} \rangle$ *is called the inner product of the vector* $\mathbf{u}$ *with the vector* $\mathbf{v}$.

$\langle \cdot, \cdot \rangle$ *is called the inner product in* $\mathbb{R}^n$.

Remark. Many people refer to $\langle , \rangle$ as the 'dot product'.

A common alternative notation for $\langle \mathbf{u}, \mathbf{v} \rangle$ is $\mathbf{u} \bullet \mathbf{v}$. For this reason, we may, for convenience, read it as '$\mathbf{u}$ dot $\mathbf{v}$'.

2. **Theorem (1). (Basic properties of inner product.)**

The statements below hold:

(a) *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
Then $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

(b) *Suppose* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$.
Then $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$.

(c) *Suppose* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$.
Then $\langle \mathbf{w}, \alpha\mathbf{u} + \beta\mathbf{v} \rangle = \alpha \langle \mathbf{w}, \mathbf{u} \rangle + \beta \langle \mathbf{w}, \mathbf{v} \rangle$.

(d) *Suppose* $\mathbf{u} \in \mathbb{R}^n$.
Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$.
Moreover, equality holds if and only if $\mathbf{u} = \mathbf{0}_n$.

3. **Proof of Theorem (1).**

(a) Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Denote the j-th entry of $\mathbf{u}, \mathbf{v}$ as u_j, v_j respectively.
Then $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$.
Similarly, $\langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{v}^t\mathbf{u} = v_1u_1 + v_2u_2 + \cdots + v_nu_n$.
Therefore $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

(b) Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.
Then
$\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = (\alpha\mathbf{u} + \beta\mathbf{v})^t\mathbf{w} = (\alpha\mathbf{u}^t + \beta\mathbf{v}^t)\mathbf{w} = \alpha\mathbf{u}^t\mathbf{w} + \beta\mathbf{v}^t\mathbf{w} = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$.

(c) Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.
Then $\langle \mathbf{w}, \alpha\mathbf{u} + \beta\mathbf{v} \rangle = \langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{w}, \mathbf{u} \rangle + \beta\langle \mathbf{w}, \mathbf{v} \rangle$.

(d) Suppose $\mathbf{u} \in \mathbb{R}^n$. Denote the j-th entry of $\mathbf{u}$ as u_j respectively.
Then $\langle \mathbf{u}, \mathbf{u} \rangle = {u_1}^2 + {u_2}^2 + \cdots + {u_n}^2 \geq 0$.
$\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $u_j = 0$ for each $j = 1, 2, \cdots, n$.
The latter happens exactly when $\mathbf{u} = \mathbf{0}_n$.

4. **Definition. (Norm.)**

For any $\mathbf{u} \in \mathbb{R}^n$*, the number* $\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ *is called the norm of the vector* $\mathbf{u}$*, and is denoted by* $\|\mathbf{u}\|$.

Remark. By definition, $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$. It is often read as 'norm-square of u'.

5. **Theorem (2). (Basic properties of norm.)**

The statements below hold:

(a) *Suppose* $\mathbf{u} \in \mathbb{R}^n$.
Then $\|\mathbf{u}\| \geq 0$.
Moreover, equality holds if and only if $\mathbf{u} = \mathbf{0}_n$.

(b) *Suppose* $\mathbf{u} \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$.
Then $\|\alpha \mathbf{u}\| = |\alpha| \cdot \|\mathbf{u}\|$.

6. **Proof of Theorem (2).**

(a) Suppose $\mathbf{u} \in \mathbb{R}^n$. By definition, $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle \geq 0$. Then $\|\mathbf{u}\| \geq 0$.
Moreover, $\|\mathbf{u}\| = 0$ if and only if $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. The latter happens exactly when $\mathbf{u} = \mathbf{0}_n$.

(b) Suppose $\mathbf{u} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then $\|\alpha \mathbf{u}\|^2 = \langle \alpha \mathbf{u}, \alpha \mathbf{u} \rangle = \alpha^2 \langle \mathbf{u}, \mathbf{u} \rangle = \alpha^2 \|\mathbf{u}\|^2$.
Therefore $\|\alpha \mathbf{u}\| = |\alpha| \cdot \|\mathbf{u}\|$.

7. **Theorem (3). (Conversion between inner product and norm.)**

The statements below hold:

(a) *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
Then $\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \pm 2 \langle \mathbf{u}, \mathbf{v} \rangle$ *respectively.*

(b) *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
Then $\langle \mathbf{u}, \mathbf{v} \rangle = \mp \frac{1}{2}(\|\mathbf{u} \pm \mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2)$ *respectively.*

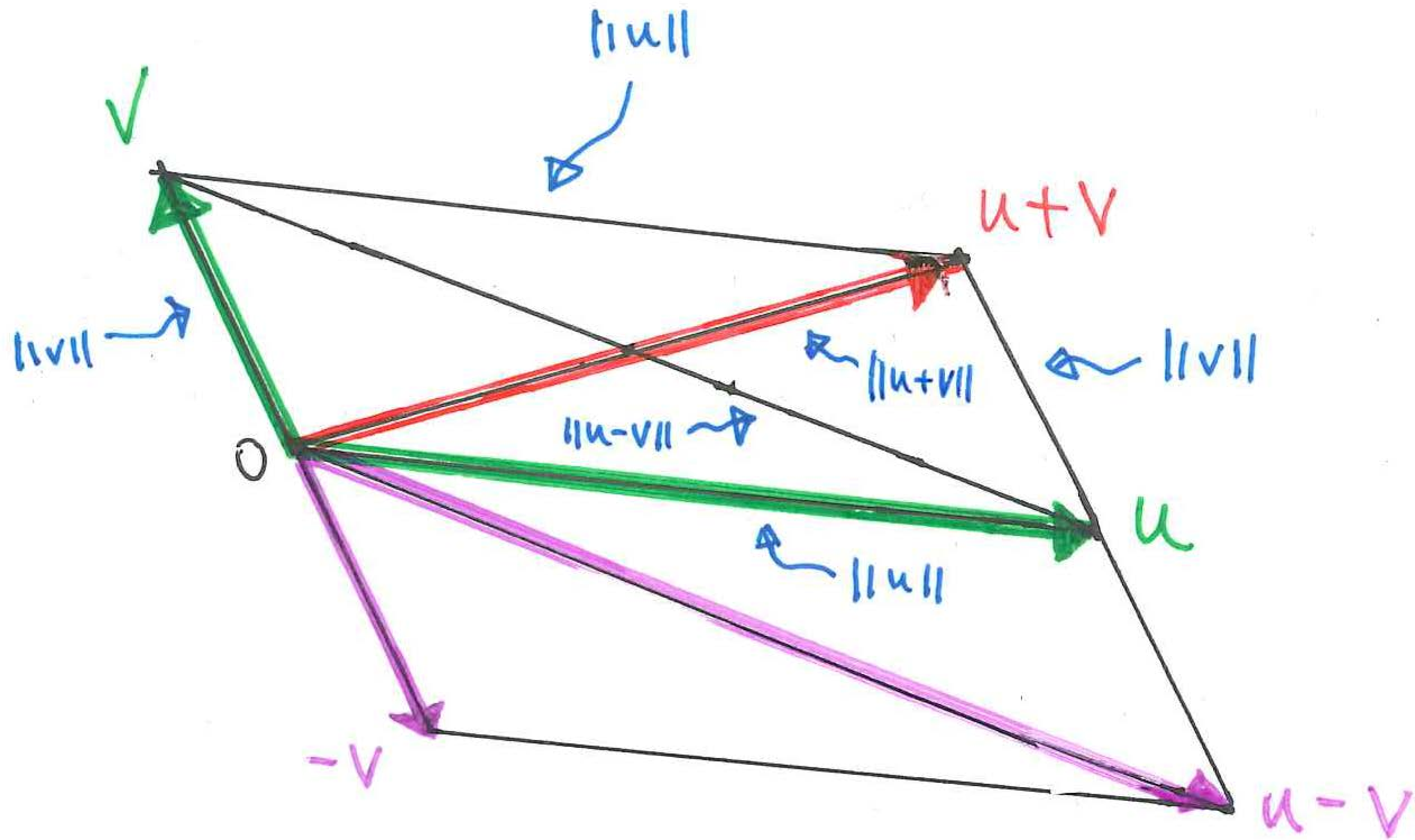
(c) *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
Then $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

Remark.

The result described in Item (b) is known as the polarization identity.

The result described in Item (c) is known as the parallelogramic identity.

Proof of Theorem (3). Exercise.



||u||
v
u+v
	v	
	u+v	
	v	
	u-v	
0
u
	u	
-v
u-v

8. **Theorem (4). (Cauchy-Schwarz Inequality.)**

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Then $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$.

Moreover, equality holds if and only if $\mathbf{u}, \mathbf{v}$ *are linearly dependent.*

Theorem (5). (Triangle Inequality.)

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Then $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

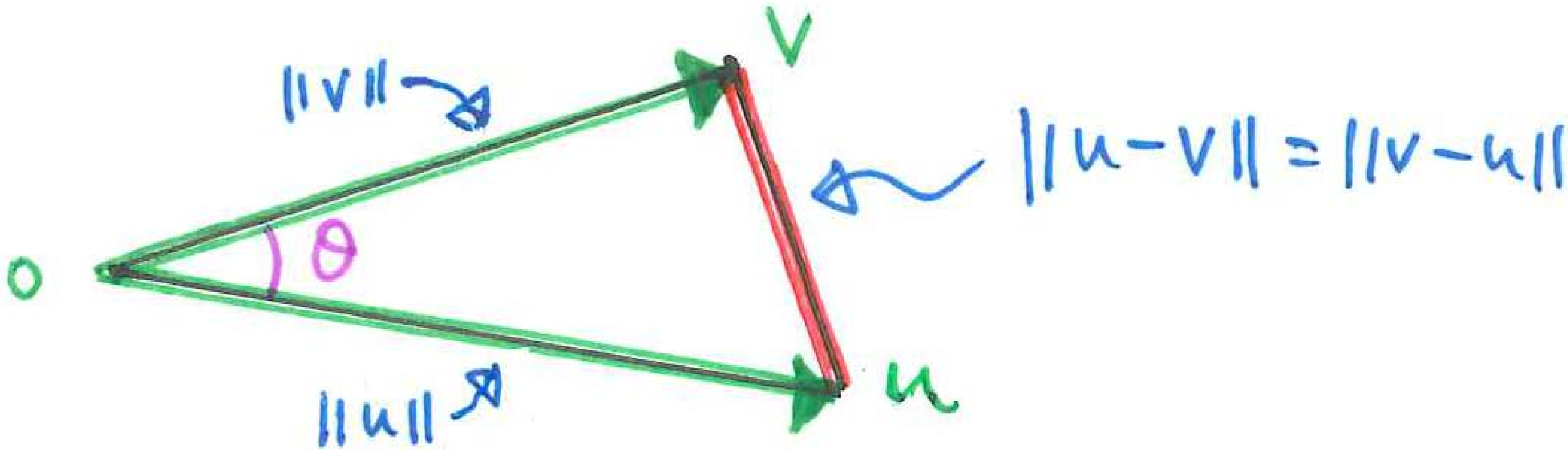
Moreover, equality holds if and only if $\mathbf{u}, \mathbf{v}$ *are non-negative scalar multiples of each other.*

Corollary to Theorem (5). (Triangle Inequality also.)

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Then $\|\mathbf{u} - \mathbf{v}\| \geq \Big| \|\mathbf{u}\| - \|\mathbf{v}\| \Big|$.

Moreover, equality holds if and only if $\mathbf{u}, \mathbf{v}$ *are non-negative scalar multiples of each other.*



v
||v||
||u-v|| = ||v-u||
0
θ
||u||
u

9. **Definition. (Orthogonality.)**

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

We say $\mathbf{u}$ *is orthogonal to* $\mathbf{v}$, *and write* $\mathbf{u} \perp \mathbf{v}$, *if and only if* $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

10. **Theorem (6). (Basic properties of orthogonality.)**

The statements below hold:

(a) *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
Then $\mathbf{u} \perp \mathbf{v}$ *if and only if* $\mathbf{v} \perp \mathbf{u}$.

(b) *Suppose* $\mathbf{u} \in \mathbb{R}^n$.
Then $\mathbf{u} \perp \mathbf{u}$ *if and only if* $\mathbf{u} = \mathbf{0}_n$.

(c) *Suppose* $\mathbf{u} \in \mathbb{R}^n$.
Then ($\mathbf{u} \perp \mathbf{v}$ *for any* $\mathbf{v} \in \mathbb{R}^n$) *if and only if* $\mathbf{u} = \mathbf{0}_n$.

(d) *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
Then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ *if and only if* $\mathbf{u} \perp \mathbf{v}$.

Proof of Theorem (6). Exercise (in the matrix/vector algebra).