1. **Lemma (1).**

   Let H, B *be* $(n \times n)$*-square matrices. Suppose* H *is a row-operation matrix.*
   Then $\det(HB) = \det(H)\det(B)$.

   Proof of Lemma (1).

   Let H, B be $(n \times n)$-square matrices. Suppose H is a row-operation matrix.

   (a) Suppose H is the row operation matrix corresponding to the row operation $\alpha R_i + R_k$ for some distinct i, k and for some real number α.
   Then $\det(H) = 1$.
   HB is obtained by B by adding a scalar multiple of the i-th row to the k-th row.
   Then $\det(HB) = \det(B)$.
   Therefore $\det(HB) = 1 \cdot \det(B) = \det(H)\det(B)$.

(b) Suppose H is the row operation matrix corresponding to the row operation βR_i for some non-zero real number β.
Then $\det(H) = \beta$.
HB is obtained by B by multiplying every entry of the i-th row by β.
Then $\det(HB) = \beta \det(B)$.
Therefore $\det(HB) = beta \det(B) = \det(H) \det(B)$.

(c) Suppose H is the row operation matrix corresponding to the row operation $R_i \leftrightarrow R_k$ for some distinct i, k.
Then $\det(H) = -1$.
HB is obtained by B by interchanging the i-th row and the k-th row.
Then $\det(HB) = -\det(B)$.
Therefore $\det(HB) = -\det(B) = \det(H) \det(B)$.

Hence, in any case, $\det(HB) = \det(H) \det(B)$.

2. **Theorem (2).**

Let A, B be $(n \times n)$-square matrices. Suppose A is nonsingular. Then $\det(AB) = \det(A)\det(B)$.

Proof of Theorem (2).

Let A, B be $(n \times n)$-square matrices. Suppose A is nonsingular.

Then there are some k row-operation matrices, say, $H_1, H_2, \cdots, H_k$, so that

$$A = H_k H_{k-1} \cdots H_2 H_1.$$

Therefore

$$\begin{aligned}
\det(AB) &= \det(H_k H_{k-1} \cdots H_2 H_1 B) \\
&= \det(H_k)\det(H_{k-1} \cdots H_2 H_1 B) \\
&= \cdots \\
&= \det(H_k)\det(H_{k-1}) \cdots \det(H_2)\det(H_1 B) \\
&= \det(H_k)\det(H_{k-1}) \cdots \det(H_2)\det(H_1)\det(B) \\
&= \det(H_k H_{k-1} \cdots H_2 H_1)\det(B) = \det(A)\det(B)
\end{aligned}$$

Then $\det(AB) = \det(A)\det(B)$.

3. **Lemma (3).**

Let C be an $(n \times n)$-square matrix. Suppose C is singular.

Then $\det(C) = 0$.

Proof of Lemma (3).

Let C be an $(n \times n)$-square matrix. Suppose C is singular.

Denote by C' the reduced row-echelon form which is row-equivalent to C.

Note that $\det(C') = 0$, because there is at least one entire row of 0's in C'.

There is some non-singular $(n \times n)$-square matrix A such that $C = AC'$. (Why?)

Then $\det(C) = \det(AC') = \det(A)\det(C') = 0$.

4. **Theorem (4).**

Let A, B be $(n \times n)$-square matrices. Suppose A is singular.

Then $\det(AB) = 0 = \det(A)\det(B)$.

Proof of Theorem (4).

Let A, B be $(n \times n)$-square matrices. Suppose A is singular.

Then $\det(A) = 0$. Therefore $\det(A)\det(B) = 0$.

Since A is singular, AB is also singular. Then $\det(AB) = 0$.

Therefore $\det(AB) = 0 = \det(A)\det(B)$.

5. Combining Theorem (2) and Theorem (4), we obtain the result below:

Theorem (ζ).
Suppose A, B are $(n \times n)$-square matrices.
Then $\det(AB) = \det(A)\det(B)$.

Remark. Actually it further follows that

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA).$$

However, note that AB and BA are not necessarily the same matrix.

6. An immediate consequence of Theorem (ζ) is Theorem (η).

Theorem (η).
Suppose A is an $(n \times n)$-square matrix. Then the statements below hole:

(a) *For any positive integer p,* $\det(A^p) = (\det(A))^p$.

(b) *Suppose A is invertible. Then* $\det(A) \neq 0$, *and* $\det(A^{-1}) = (\det(A))^{-1}$.

7. Statement (b) in Theorem (η) tells us that if a square matrix is invertible then its determinant is non-zero.

It is natural to ask whether it is true that if the determinant of a square matrix is non-zero then the matrix concerned is invertible. The answer is provided by Theorem (5).

Theorem (5).
Let A be an $(n \times n)$-square matrix. Suppose $\det(A) \neq 0$. *Then A is invertible.*

Proof of Theorem (5).
Let A be an $(n \times n)$-square matrix. Suppose $\det(A) \neq 0$.
Denote by A' the reduced row-echelon form which is row-equivalent to A.
There exists some non-singular $(n \times n)$-square matrix H such that $A' = HA$.
By Theorem (ζ), we have $\det(A') = \det(H)\det(A)$.
Since H is non-singular, we have $\det(H) \neq 0$. By assumption, $\det(A) \neq 0$.
Then $\det(A') \neq 0$.
By assumption A' is a reduced row-echelon form. Since $\det(A') \neq 0$, there is no row of A' which is a row of 0's. Then every row of A' contains a leading one.
Therefore $A' = I_n$.
Hence A is row equivalent to I_n. Then A is non-singular.

8. Combining Theorem (η) and Theorem (5), we obtain the result below:

Theorem (θ).
Suppose A is an $(n \times n)$-square matrix.
Then the statements below are logically equivalent:

(a) *A is non-singular.*
(b) *A is invertible.*
(c) $\det(A) \neq 0$.

9. **Corollary to Theorem (θ).**
Suppose A is an $(n \times n)$-square matrix.
Then the statements below are logically equivalent:

(a) *A is singular.*
(b) *A is not invertible.*
(c) $\det(A) = 0$.

10. We now compile and re-organized all the various re-formulations for the notions of non-singularity and invertibility that we have learnt so far into one single result:

Theorem (ι). (Various re-formulations for the notions of non-singularity and invertibility.)
Let A be an $(n \times n)$-matrix.

(a) *The statements below are logically equivalent:*

   i. *A is non-singular.*
   ii. *For any vector $\mathbf{v}$ in $\mathbb{R}^n$, if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.*
   iii. *The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.*
   iv. *A is row-equivalent to I_n.*
   v. *A is invertible.*
   vi. *There exists some $(n \times n)$-square matrix H such that $HA = I_n$.*
   vii. *There exists some $(n \times n)$-square matrix G such that $AG = I_n$.*
   viii. *For any vector $\mathbf{b}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{b})$ has one and only one solution, namely, '$\mathbf{x} = A^{-1}\mathbf{b}$'.*
   ix. *For any vector $\mathbf{c}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{c})$ has at least one solution.*
   x. *For any vector $\mathbf{d}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{d})$ has at most one solution.*

(b) *The statements below are logically equivalent:*

i. A *is non-singular.*

ii. A^t *is non-singular.*

iii. *For any vector* $\mathbf{v}$ *in* $\mathbb{R}^n$, *if* $A^t\mathbf{v} = \mathbf{0}$ *then* $\mathbf{v} = \mathbf{0}$.

iv. *The trivial solution is the only solution of the homogeneous system* $\mathcal{LS}(A^t, \mathbf{0})$.

v. A^t *is row-equivalent to* I_n.

vi. A^t *is invertible.*

vii. *There exists some* $(n \times n)$*-square matrix* H *such that* $JA^t = I_n$.

viii. *There exists some* $(n \times n)$*-square matrix* G *such that* $A^tK = I_n$.

ix. *For any vector* $\mathbf{b}$ *in* $\mathbb{R}^n$, *the system* $\mathcal{LS}(A^t, \mathbf{b})$ *has one and only one solution, namely,* '$\mathbf{x} = (A^t)^{-1}\mathbf{b}$'.

x. *For any vector* $\mathbf{c}$ *in* $\mathbb{R}^n$, *the system* $\mathcal{LS}(A^t, \mathbf{c})$ *has at least one solution.*

xi. *For any vector* $\mathbf{d}$ *in* $\mathbb{R}^n$, *the system* $\mathcal{LS}(A^t, \mathbf{d})$ *has at most one solution.*

(c) *Denote the j-th column of A by $\mathbf{u}_j$ for each $j = 1, 2, \cdots, n$.*
The statements below are logically equivalent:

i. *A is non-singular.*
ii. *Every vector in $\mathbb{R}^n$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.*
iii. *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent.*
iv. *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$.*
v. *The dimension of the column space of A is n.*
vi. *The dimension of the null space of A is 0.*
vii. $\det(A) \neq 0$.

(d) *Denote the i-th row of A by $\mathbf{w}_i$ for each $i = 1, 2, \cdots, n$.*
The statements below are logically equivalent:

i. *A is non-singular.*
ii. *A^t is non-singular.*
iii. *Every vector in $\mathbb{R}^n$ is a linear combination of ${\mathbf{w}_1}^t, {\mathbf{w}_2}^t, \cdots, {\mathbf{w}_n}^t$.*
iv. *${\mathbf{w}_1}^t, {\mathbf{w}_2}^t, \cdots, {\mathbf{w}_n}^t$. are linearly independent.*
v. *${\mathbf{w}_1}^t, {\mathbf{w}_2}^t, \cdots, {\mathbf{w}_n}^t$. constitute a basis for $\mathbb{R}^n$.*
vi. *The dimension of the row space of A is n.*
vii. *The dimension of the null space of A^t is 0.*
viii. $\det(A^t) \neq 0$.

(e) *Now further suppose A is non-singular, with a sequence of row operations*

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} C_{p-1} \xrightarrow[\rho_{p-1}]{} C_p = I_n,$$

and with H_k being the row-operation matrix corresponding to ρ_k for each k.
Then $[I_n|A^{-1}]$ is the resultant of the application of the same sequence of row operations $\rho_1, \rho_2, \cdots, \rho_{p-1}$ starting from $[A|I_n]$:

$$\begin{aligned} [A|I_n] \ &= \ [C_1|I_n] \xrightarrow[\rho_1]{} [C_2|H_1] \\ &\xrightarrow[\rho_2]{} [C_3|H_2H_1] \\ &\xrightarrow[\rho_3]{} \cdots\cdots \\ &\xrightarrow[\rho_{p-2}]{} [C_{p-1}|H_{p-2}\cdots H_2H_1] \\ &\xrightarrow[\rho_{p-1}]{} [C_p|H_{p-1}\cdots H_2H_1] = [I_n|A^{-1}]. \end{aligned}$$

Moreover, A^{-1} and A are respectively given as products of row-operation matrices by

$$A^{-1} = H_{p-1}\cdots H_2H_1, \qquad\qquad A = {H_1}^{-1}{H_2}^{-1}\cdots {H_{p-1}}^{-1}.$$