MATH1030 Determinants.

1. **Definition. (Submatrices of a square matrix)**

   Let A be an $(n \times n)$-square matrix.

   For each k, ℓ, the (k, ℓ)-th submatrix of A is defined to be the $((n-1) \times (n-1))$-matrix resultant from simultaneously deleted the k-th row and ℓ-th column of A. It is denoted by $A(k|\ell)$.

2. **Illustration.**

   (a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.
   Then $A(1|1) = [a_{22}]$, $A(1|2) = [a_{21}]$, $A(2|1) = [a_{12}]$, $A(2|2) = [a_{11}]$.

   (b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

   Then

$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \quad A(1|2) = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad A(1|3) = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},$$
$$A(2|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}, \quad A(2|2) = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}, \quad A(2|3) = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix},$$
$$A(3|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}, \quad A(3|2) = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}, \quad A(3|3) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

3. **Definition. ('Inductive definition for determinants' through 'expansion' along the first column.)**

   Let A be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}.

   (a) *Suppose $n = 1$. Then we define the determinant of A, which is denoted by* $\det(A)$, *to be the number which is the only entry of A.*

   (b) *Suppose $n > 1$. Then we define the determinant of A, which is denoted by* $\det(A)$, *by*

$$\det(A) = (-1)^{1+1}a_{11}\det(A(1|1)) + (-1)^{2+1}a_{21}\det(A(2|1)) + (-1)^{3+1}a_{31}\det(A(3|1)) + \cdots + (-1)^{n+1}a_{n1}\det(A(n|1)).$$

   Remark. The 'formula'

$$\det(A) = (-1)^{1+1}a_{11}\det(A(1|1)) + (-1)^{2+1}a_{21}\det(A(2|1)) + (-1)^{3+1}a_{31}\det(A(3|1)) + \cdots + (-1)^{n+1}a_{n1}\det(A(n|1))$$

   is usually referred to as the 'expansion' of a determinant along the first column.

4. **Illustration.**

   (a) Suppose $A = [a_{11}]$. Then $\det(A) = a_{11}$.

   (b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We have $A(1|1) = [a_{22}]$ and $A(2|1) = [a_{12}]$.
   Then $\det(A) = a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) = a_{11}a_{22} - a_{12}a_{21}$.

   (c) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

   We have $A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$, $A(2|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$, and $A(3|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$.

   Then

$$\begin{aligned} \det(A) &= a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) + a_{31}\det(A(3|1)) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

   (d) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$.

   We have $A(1|1) = \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$, $A(2|1) = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$, $A(3|1) = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$, and
   $A(4|1) = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix}$.

Then

$$\begin{aligned}
\det(A) &= a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) + a_{31}\det(A(3|1)) - a_{41}\det(A(4|1)) \\
&= \cdots \\
&= a_{11}a_{22}a_{33}a_{44} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} \\
&\quad + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} + a_{12}a_{23}a_{31}a_{44} \\
&\quad + a_{13}a_{24}a_{31}a_{42} + a_{13}a_{21}a_{32}a_{44} + a_{13}a_{22}a_{34}a_{41} \\
&\quad + a_{14}a_{23}a_{32}a_{41} + a_{14}a_{22}a_{31}a_{43} + a_{14}a_{21}a_{33}a_{42} \\
&\quad - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{11}a_{23}a_{32}a_{44} \\
&\quad - a_{12}a_{21}a_{33}a_{44} - a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43} \\
&\quad - a_{13}a_{24}a_{32}a_{41} - a_{13}a_{22}a_{31}a_{44} - a_{13}a_{21}a_{34}a_{42} \\
&\quad - a_{14}a_{23}a_{31}a_{42} - a_{14}a_{21}a_{32}a_{43} - a_{14}a_{22}a_{33}a_{41}
\end{aligned}$$

5. **Examples.**

   (a) $\det(\begin{bmatrix} 1 & 7 \\ 6 & 9 \end{bmatrix}) = 1\cdot 9 - 6\cdot 7 = -33.$

   (b) $\det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) = 1\cdot\det(\begin{bmatrix} 9 & 8 \\ 1 & 5 \end{bmatrix}) - 6\cdot\det(\begin{bmatrix} 7 & 0 \\ 1 & 5 \end{bmatrix}) + 0\cdot\det(\begin{bmatrix} 7 & 0 \\ 9 & 8 \end{bmatrix}) = 1\cdot(9\cdot 5 - 1\cdot 8) - 6(7\cdot 5 - 1\cdot 0) = -173.$

   (c)

$$\begin{aligned}
&\det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\
= \;& 1\cdot\det(\begin{bmatrix} 5 & 2 & 5 \\ 9 & 8 & 0 \\ 9 & 8 & 3 \end{bmatrix}) - 0\cdot\det(\begin{bmatrix} 9 & 7 & 7 \\ 9 & 8 & 0 \\ 9 & 8 & 3 \end{bmatrix}) + 1\cdot\det(\begin{bmatrix} 9 & 7 & 7 \\ 5 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}) - 1\cdot\det(\begin{bmatrix} 9 & 7 & 7 \\ 5 & 2 & 5 \\ 9 & 8 & 0 \end{bmatrix}) \\
= \;& \Big(5\cdot\det(\begin{bmatrix} 8 & 0 \\ 8 & 3 \end{bmatrix}) - 9\cdot\det(\begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix}) + 9\cdot\det(\begin{bmatrix} 2 & 5 \\ 8 & 0 \end{bmatrix})\Big) - 0 \\
& + \Big(9\cdot\det(\begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix}) - 5\cdot\det(\begin{bmatrix} 7 & 7 \\ 8 & 3 \end{bmatrix}) + 9\cdot\det(\begin{bmatrix} 7 & 7 \\ 2 & 5 \end{bmatrix})\Big) \\
& - \Big(9\cdot\det(\begin{bmatrix} 2 & 5 \\ 8 & 0 \end{bmatrix}) - 5\cdot\det(\begin{bmatrix} 7 & 7 \\ 8 & 0 \end{bmatrix}) + 9\cdot\det(\begin{bmatrix} 7 & 7 \\ 2 & 5 \end{bmatrix})\Big) \\
= \;& [5\cdot 24 - 9\cdot(-34) + 9\cdot(-40)] - 0 + [9\cdot(-34) - 5\cdot(-35) + 9\cdot 21] - [9\cdot(-40) - 5\cdot(-56) + 9\cdot(21)] \\
= \;& 15
\end{aligned}$$

6. **Theorem (1). ('Expansion' of a determinant along any arbitrary column.)**

   Let A be an $(n\times n)$-square matrix, whose (i,j)-th entry is denoted by a_{ij}.

   Suppose $n > 1$. Then, for each $j = 1, 2, \cdots, n$,

$$\det(A) = (-1)^{1+j}a_{1j}\det(A(1|j)) + (-1)^{2+j}a_{2j}\det(A(2|j)) + (-1)^{3+j}a_{3j}\det(A(3|j)) + \cdots + (-1)^{n+j}a_{nj}\det(A(n|j)).$$

   Proof of Theorem (1). Omitted. (This can be done with mathematical induction.)

7. **Illustration.**

   (a) Suppose A is a (3×3)-square matrix, whose (i,j)-th entry is denoted by a_{ij}. Then

$$\begin{array}{rcrcrcr}
\det(A) &=& a_{11}\det(A(1|1)) &-& a_{21}\det(A(2|1)) &+& a_{31}\det(A(3|1)), \\
\det(A) &=& -a_{12}\det(A(1|2)) &+& a_{22}\det(A(2|2)) &-& a_{32}\det(A(3|2)), \\
\det(A) &=& a_{13}\det(A(1|3)) &-& a_{23}\det(A(2|3)) &+& a_{33}\det(A(3|3)).
\end{array}$$

   (b) Suppose A is a (4×4)-square matrix, whose (i,j)-th entry is denoted by a_{ij}. Then

$$\begin{array}{rcrcrcrcr}
\det(A) &=& a_{11}\det(A(1|1)) &-& a_{21}\det(A(2|1)) &+& a_{31}\det(A(3|1)) &-& a_{41}\det(A(4|1)), \\
\det(A) &=& -a_{12}\det(A(1|2)) &+& a_{22}\det(A(2|2)) &-& a_{32}\det(A(3|2)) &+& a_{42}\det(A(4|2)), \\
\det(A) &=& a_{13}\det(A(1|3)) &-& a_{23}\det(A(2|3)) &+& a_{33}\det(A(3|3)) &-& a_{43}\det(A(4|3)), \\
\det(A) &=& -a_{14}\det(A(1|4)) &+& a_{24}\det(A(2|4)) &-& a_{34}\det(A(3|4)) &+& a_{44}\det(A(4|4)).
\end{array}$$

8. **Examples.**

(a) $\det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) = -7 \cdot \det(\begin{bmatrix} 6 & 8 \\ 0 & 5 \end{bmatrix}) + 9 \cdot \det(\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}) - 1 \cdot \det(\begin{bmatrix} 1 & 0 \\ 6 & 8 \end{bmatrix}) = \cdots = -173.$

$\det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) = 0 \cdot \det(\begin{bmatrix} 6 & 9 \\ 0 & 1 \end{bmatrix}) - 8 \cdot \det(\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}) + 5 \cdot \det(\begin{bmatrix} 1 & 7 \\ 6 & 9 \end{bmatrix}) = \cdots = -173.$

(b)

$$\begin{aligned} & \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\ = & -9 \cdot \det(\begin{bmatrix} 0 & 2 & 5 \\ 1 & 8 & 0 \\ 1 & 8 & 3 \end{bmatrix}) + 5 \cdot \det(\begin{bmatrix} 1 & 7 & 7 \\ 1 & 8 & 0 \\ 1 & 8 & 3 \end{bmatrix}) - 9 \cdot \det(\begin{bmatrix} 1 & 7 & 7 \\ 0 & 2 & 5 \\ 1 & 8 & 3 \end{bmatrix}) + 9 \cdot \det(\begin{bmatrix} 1 & 7 & 7 \\ 0 & 2 & 5 \\ 1 & 8 & 0 \end{bmatrix}) = \cdots = 15 \end{aligned}$$

$$\begin{aligned} & \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\ = & 7 \cdot \det(\begin{bmatrix} 0 & 5 & 5 \\ 1 & 9 & 0 \\ 1 & 9 & 3 \end{bmatrix}) - 2 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 1 & 9 & 0 \\ 1 & 9 & 3 \end{bmatrix}) + 8 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 5 \\ 1 & 9 & 3 \end{bmatrix}) - 8 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 5 \\ 1 & 9 & 0 \end{bmatrix}) = \cdots = 15 \end{aligned}$$

$$\begin{aligned} & \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\ = & -7 \cdot \det(\begin{bmatrix} 0 & 5 & 2 \\ 1 & 9 & 8 \\ 1 & 9 & 8 \end{bmatrix}) + 5 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 1 & 9 & 8 \\ 1 & 9 & 8 \end{bmatrix}) - 0 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 2 \\ 1 & 9 & 8 \end{bmatrix}) + 3 \cdot \det(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 2 \\ 1 & 9 & 8 \end{bmatrix}) = \cdots = 15 \end{aligned}$$

9. **Theorem (2). ('Expansion' of a determinant along the first row.)**

Let A be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}.

Suppose $n > 1$. Then

$$\det(A) = (-1)^{1+1} a_{11} \det(A(1|1)) + (-1)^{1+2} a_{12} \det(A(1|2)) + (-1)^{1+3} a_{13} \det(A(1|3)) + \cdots + (-1)^{1+n} a_{1n} \det(A(1|n)).$$

Proof of Theorem (2). Omitted. (This can be done with mathematical induction.)

10. **Illustration.**

(a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We have $A(1|1) = [a_{22}]$ and $A(2|1) = [a_{12}]$.

Then $\det(A) = a_{11} \det(A(1|1)) - a_{12} \det(A(1|2)) = a_{11}a_{22} - a_{12}a_{21}$.

(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

We have $A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$, $A(1|2) = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$, and $A(1|3) = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

Then

$$\begin{aligned} \det(A) &= a_{11} \det(A(1|1)) - a_{12} \det(A(1|2)) + a_{13} \det(A(1|3)) \\ &= \cdots \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

(c) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$.

We have $A(1|1) = \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$, $A(1|2) = \begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix}$, $A(1|3) = \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}$, and

$A(1|4) = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$.

Then

$$\begin{aligned}\det(A) &= a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) + a_{13}\det(A(1|3)) - a_{14}\det(A(1|4))\\ &= \cdots\\ &= a_{11}a_{22}a_{33}a_{44} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43}\\ &\quad +a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} + a_{12}a_{23}a_{31}a_{44}\\ &\quad +a_{13}a_{24}a_{31}a_{42} + a_{13}a_{21}a_{32}a_{44} + a_{13}a_{22}a_{34}a_{41}\\ &\quad +a_{14}a_{23}a_{32}a_{41} + a_{14}a_{22}a_{31}a_{43} + a_{14}a_{21}a_{33}a_{42}\\ &\quad -a_{11}a_{22}a_{34}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{11}a_{23}a_{32}a_{44}\\ &\quad -a_{12}a_{21}a_{33}a_{44} - a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43}\\ &\quad -a_{13}a_{24}a_{32}a_{41} - a_{13}a_{22}a_{31}a_{44} - a_{13}a_{21}a_{34}a_{42}\\ &\quad -a_{14}a_{23}a_{31}a_{42} - a_{14}a_{21}a_{32}a_{43} - a_{14}a_{22}a_{33}a_{41}\end{aligned}$$

11. **Theorem (α).**

    Suppose A be a square matrix. Then $\det(A^t) = \det(A)$.

    Proof of Theorem (α). Omitted. (This can be done with mathematical induction. Apply the definition and Theorem (2).)

12. **Theorem (3). ('Expansion' of a determinant along any arbitrary row.)**

    Let A be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}.

    Suppose $n > 1$. Then, for each $i = 1, 2, \cdots, n$,

    $$\det(A) = (-1)^{i+1}a_{i1}\det(A(i|1)) + (-1)^{i+2}a_{i2}\det(A(i|2)) + (-1)^{i+3}a_{i3}\det(A(i|3)) + \cdots + (-1)^{i+n}a_{in}\det(A(i|n)).$$

    Proof of Theorem (3). This is a consequence of Theorem (1) and Theorem (α) combined.

13. **Illustration.**

    (a) Suppose A is a (3×3)-square matrix, whose (i, j)-th entry is denoted by a_{ij}. Then

    $$\begin{array}{rcrcrcr}\det(A) &=& a_{11}\det(A(1|1)) &-& a_{12}\det(A(1|2)) &+& a_{13}\det(A(1|3)),\\ \det(A) &=& -a_{21}\det(A(2|1)) &+& a_{22}\det(A(2|2)) &-& a_{23}\det(A(2|3)),\\ \det(A) &=& a_{31}\det(A(3|1)) &-& a_{32}\det(A(3|2)) &+& a_{33}\det(A(3|3)).\end{array}$$

    (b) Suppose A is a (4×4)-square matrix, whose (i, j)-th entry is denoted by a_{ij}. Then

    $$\begin{array}{rcrcrcrcr}\det(A) &=& a_{11}\det(A(1|1)) &-& a_{12}\det(A(1|2)) &+& a_{13}\det(A(1|3)) &-& a_{14}\det(A(1|4)),\\ \det(A) &=& -a_{21}\det(A(2|1)) &+& a_{22}\det(A(2|2)) &-& a_{23}\det(A(2|3)) &+& a_{24}\det(A(2|4)),\\ \det(A) &=& a_{31}\det(A(3|1)) &-& a_{32}\det(A(3|2)) &+& a_{33}\det(A(3|3)) &-& a_{34}\det(A(3|4)),\\ \det(A) &=& -a_{41}\det(A(4|1)) &+& a_{42}\det(A(4|2)) &-& a_{43}\det(A(4|3)) &+& a_{44}\det(A(4|4)).\end{array}$$

14. **Key theoretical examples.**

    (a) Let A be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}.

    i. Suppose there is some q amongst $1, 2, \cdots, n$ for which the q-th column is all zero.
    Then

    $$\begin{aligned}\det(A) &= (-1)^{1+q}a_{1q}\det(A(1|q)) + (-1)^{2+q}a_{2q}\det(A(2|q)) + (-1)^{3+q}a_{3q}\det(A(3|q))\\ &\quad + \cdots + (-1)^{n+q}a_{nq}\det(A(n|q))\\ &= 0.\end{aligned}$$

    ii. Suppose there is some p amongst $1, 2, \cdots, n$ for which the p-th row is all zero.
    Then

    $$\begin{aligned}\det(A) &= (-1)^{p+1}a_{p1}\det(A(p|1)) + (-1)^{p+2}a_{p2}\det(A(p|2)) + (-1)^{p+3}a_{p3}\det(A(p|3))\\ &\quad + \cdots + (-1)^{p+n}a_{pn}\det(A(p|n))\\ &= 0.\end{aligned}$$

    (b) Let D be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by d_{ij}.
    Suppose $d_{ij} = 0$ whenever $i \neq j$. (Such a matrix is called a diagonal matrix.)
    We have $\det(D) = d_{11}\det(D(1|1))$.
    Note that $D(1|1)$ is itself a diagonal matrix.
    Then $\det(D) = d_{11}d_{22}d_{33}\cdots d_{nn}$.
    Remark. In particular, $\det(I_n) = 1$.

(c) i. Let A be an $(n \times n)$-square matrix, whose (i,j)-th entry is denoted by a_{ij}.
Suppose $a_{ij} = 0$ whenever $i > j$. (Such a matrix is called an upper-triangular matrix.)

A is explicitly given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & \cdots & a_{3n} \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & & a_{nn} \end{bmatrix}$.

By definition, we have $\det(A) = a_{11}\det(A(1|1))$.

Note that $A(1|1) = \begin{bmatrix} a_{22} & a_{23} & a_{24} & \cdots & \cdots & a_{2n} \\ 0 & a_{33} & a_{34} & \cdots & \cdots & a_{3n} \\ 0 & 0 & a_{44} & \cdots & \cdots & a_{4n} \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & & a_{nn} \end{bmatrix}$. (So $A(1|1)$ is also an upper triangular matrix.)

Then, 'inductively', we have $\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}$.

ii. Let B be an $(n \times n)$-square matrix, whose (i,j)-th entry is denoted by b_{ij}.
Suppose $b_{ij} = 0$ whenever $i < j$. (Such a matrix is called a lower-triangular matrix.)

B is explicitly given by $B = \begin{bmatrix} b_{11} & 0 & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \\ b_{n1} & b_{n2} & b_{n3} & \cdots & \cdots & b_{nn} \end{bmatrix}$.

Note that B^t is an upper triangular matrix.
Then $\det(B) = \det(B^t) = b_{11}b_{22}b_{33}\cdots b_{nn}$.

(d) i. Denote by ρ the row operation $\alpha R_i + R_k$ on matrices with n rows.
The row operation matrix $M[\rho]$ corresponding to ρ is the $(n\times n)$-square matrix given by $M[\rho] = I_n + \alpha E_{k,i}^{n,n}$.
Note that $M[\rho]$ is an upper triangular matrix or a lower triangular matrix.
Then $\det(M[\rho]) = 1$.

ii. Denote by σ the row operation βR_i on matrices with n rows.
The row operation matrix $M[\sigma]$ corresponding to σ is given by $M[\sigma] = I_n + (\beta - 1)E_{i,i}^{n,n}$.
Note that $M[\sigma]$ is an upper triangular matrix, whose diagonal entries are made of $n-1$ copies of 1 and 1 copy of β.
Then $\det(M[\sigma]) = \beta$.

iii. Denote by τ the row operation $R_i \leftrightarrow R_k$ on matrices with n rows.
The row operation matrix $M[\tau]$ corresponding to τ is given by $M[\tau] = I_n - E_{i,i}^{n,n} - E_{k,k}^{n,n} + E_{k,i}^{n,n} + E_{i,k}^{n,n}$.
$M[\tau]$ is a symmetric matrix with $n-2$ entries of 1, along the diagonal, and 2 entries of 1 off diagonal.
Repeatedly applying the definition, we have $\det(M[\tau]) = \det(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = -1$.

(e) Let A be an $(n \times n)$-square matrix. Suppose A is a reduced row-echelon form.

i. Suppose A is non-singular. Then $A = I_n$. Therefore $\det(A) = 1$.

ii. Suppose A is singular. Then A has at least one entire row of 0's. Then $\det(A) = 0$.