MATH1030 Diagonalizability.

1. **Definition. (Diagonal matrix.)**

   Let $D$ be a $(n \times n)$-square matrix, whose $(i, j)$-th entry is denoted by $d_{ij}$. The matrix $D$ is said to be a diagonal matrix if and only if $d_{ij} = 0$ whenever $i \neq j$.

   **Remark on notation.** When $d_{11} = \alpha_1, d_{22} = \alpha_2, \cdots, d_{nn} = \alpha_n$, we may write $D = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

2. **Definition. (Diagonalizability and diagonalization.)**

   Let $A$ be an $(n \times n)$-square matrix.

   (a) Suppose $U$ is a non-singular $(n \times n)$-square matrix. Then we say that $U^{-1}AU$ is a diagonalization of $A$ if and only if $U^{-1}AU$ is a diagonal matrix.

   (b) $A$ is said to be diagonalizable if and only if there is some non-singular $(n \times n)$-square matrix $T$ such that $T^{-1}AT$ is a diagonalization of $A$.

   **Remark.** A diagonalizable matrix may have various diagonalization.

3. Recall the definition for the notions of *eigenvalue* and *eigenvector* from the handout *Eigenvalues and eigenvectors*:

   Let $A$ be an $(n \times n)$-square matrix (with real entries). Let $\lambda$ be a (real) number. Let $\mathbf{v}$ be a non-zero vector with $n$ (real) entries.

   We say $\mathbf{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$ (or equivalently, $\lambda$ is an eigenvalue of $A$ with a corresponding eigenvector $\mathbf{v}$) if and only if $A\mathbf{v} = \lambda\mathbf{v}$.

4. **Theorem (C).**

   Let $A$ is an $(n \times n)$-square matrix. Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^n$, and $U = \left[\ \mathbf{u}_1\ \middle|\ \mathbf{u}_2\ \middle|\ \cdots\ \middle|\ \mathbf{u}_n\ \right]$.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$. (So $U$ is non-singular.)

   Then the statements below are logically equivalent:

   (a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of $A$, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

   (b) $U^{-1}AU$ is a diagonal matrix, given by $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

5. **Corollary to Theorem (C).**

   Let $A$ is an $(n \times n)$-square matrix.

   Suppose $A$ has $n$ pairwise distinct eigenvalues. Then $A$ is diagonalizable.

   **Proof of Corollary to Theorem (C).**

   Each of the $n$ eigenvalues of $A$ will correspond to an eigenvector. Since the eigenvalues are pairwise distinct, the $n$ corresponding eigenvectors will be linearly independent. They will then constitute a basis for $\mathbb{R}^n$.

6. **Examples of diagonalizable matrices and their diagonalizations.**

   (a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

   It happens that $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of $A$ with respective eigenvalues $1, -2$.

   Since $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of $A$ with distinct eigenvalues, they are linearly independent.

   Then $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathbb{R}^2$.

   Define $U = \left[\ \mathbf{u}_1\ \middle|\ \mathbf{u}_2\ \right]$.

   $U$ is nonsingular, and $U^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix}$.

   By direct verification, we see that $U^{-1}AU = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, as expected from theory.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of $A$ with respective eigenvalues $1, 2, 3$.

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of $A$ with pairwise distinct eigenvalues, they are linear independent.

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathbb{R}^3$.

Define $U = \begin{bmatrix} \mathbf{u}_1 & | & \mathbf{u}_2 & | & \mathbf{u}_3 \end{bmatrix}$.

$U$ is nonsingular, and $U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix}$.

By direct verification, we see that $U^{-1}AU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, as expected from theory.

(c) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of $A$ with respective eigenvalues $4, 1, 1$.

We can check that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linear independent. (Fill in the detail.)

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathbb{R}^3$.

Define $U = \begin{bmatrix} \mathbf{u}_1 & | & \mathbf{u}_2 & | & \mathbf{u}_3 \end{bmatrix}$.

$U$ is nonsingular, and $U^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix}$.

By direct verification, we see that $U^{-1}AU = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, as expected from theory.

(d) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of $A$ with respective eigenvalues $1, -1, 3, -3$.

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of $A$ with pairwise distinct eigenvalues, they are linear independent.

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ constitute a basis for $\mathbb{R}^4$.

Define $U = \begin{bmatrix} \mathbf{u}_1 & | & \mathbf{u}_2 & | & \mathbf{u}_3 & | & \mathbf{u}_4 \end{bmatrix}$.

$U$ is nonsingular, and $U^{-1} = \begin{bmatrix} 5/8 & -1/4 & 0 & -1/8 \\ 1/4 & 1/8 & -1/8 & 0 \\ 0 & 1/8 & 5/24 & 1/12 \\ 1/8 & 0 & -1/12 & 1/24 \end{bmatrix}$.

By direct verification, we see that $U^{-1}AU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$, as expected from theory.

7. **Non-examples on diagonalizability.**

(a) Let $b$ be a real number, and $A = \begin{bmatrix} b & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$u$ is an eigenvector of $A$ with eigenvalue $b$, and every eigenvector of $A$ is a non-zero scalar multiple of $\mathbf{u}$.

Then there is no basis for $\mathbb{R}^3$ which is constituted by eigenvectors of $A$.

Therefore $A$ is not diagonalizable.

(b) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

$A$ has no eigenvalues, and hence no eigenvectors.

Then there is no basis for $\mathbb{R}^4$ which is constituted by eigenvectors of $A$.

Therefore $A$ is not diagonalizable.

8. **Proof of Theorem (C).**

Let $A$ is an $(n \times n)$-square matrix. Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^n$, and $U = \begin{bmatrix} \mathbf{u}_1 & | & \mathbf{u}_2 & | & \cdots & | & \mathbf{u}_n \end{bmatrix}$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$.

- Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of $A$, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

  [Reminder: We want to verify that a diagonalization for $A$ is given by $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.]

  Then for each $j = 1, 2, \cdots, n$, we have $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$.

  Therefore

  $$
  \begin{aligned}
  AU &= \begin{bmatrix} A\mathbf{u}_1 & | & A\mathbf{u}_2 & | & \cdots & | & A\mathbf{u}_n \end{bmatrix} \\
  &= \begin{bmatrix} \lambda_1\mathbf{u}_1 & | & \lambda_2\mathbf{u}_2 & | & \cdots & | & \lambda_n\mathbf{u}_n \end{bmatrix} \\
  &= \begin{bmatrix} \lambda_1 U\mathbf{e}_1^{(n)} & | & \lambda_2 U\mathbf{e}_2^{(n)} & | & \cdots & | & \lambda_n U\mathbf{e}_n^{(n)} \end{bmatrix} \\
  &= U \begin{bmatrix} \lambda_1\mathbf{e}_1^{(n)} & | & \lambda_2\mathbf{e}_2^{(n)} & | & \cdots & | & \lambda_n\mathbf{e}_n^{(n)} \end{bmatrix} = U\,\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)
  \end{aligned}
  $$

  Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitutes a basis for $\mathbb{R}^n$, $U$ is non-singular and invertible. The matrix $U^{-1}$ is well-defined.

  Then $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, which is a diagonal matrix.

- Suppose $U^{-1}AU$ is a diagonal matrix, given by $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

  [Reminder: We want to verify that for each $j$, $\mathbf{u}_j$ is an eigenvector of $A$ with eigenvalue $\lambda_j$.]

  Then

  $$
  \begin{aligned}
  \begin{bmatrix} A\mathbf{u}_1 & | & A\mathbf{u}_2 & | & \cdots & | & A\mathbf{u}_n \end{bmatrix} &= AU = U\,\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \\
  &= U \begin{bmatrix} \lambda_1\mathbf{e}_1^{(n)} & | & \lambda_2\mathbf{e}_2^{(n)} & | & \cdots & | & \lambda_n\mathbf{e}_n^{(n)} \end{bmatrix} \\
  &= \begin{bmatrix} \lambda_1 U\mathbf{e}_1^{(n)} & | & \lambda_2 U\mathbf{e}_2^{(n)} & | & \cdots & | & \lambda_n U\mathbf{e}_n^{(n)} \end{bmatrix} \\
  &= \begin{bmatrix} \lambda_1\mathbf{u}_1 & | & \lambda_2\mathbf{u}_2 & | & \cdots & | & \lambda_n\mathbf{u}_n \end{bmatrix}
  \end{aligned}
  $$

  Therefore, for each $j = 1, 2, \cdots, n$, we have $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$.

  Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of $A$, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

9. **Lemma (1).**

   *Suppose $A$ is an $(n \times n)$-square matrix. Then $A$ is singular if and only if $0$ is an eigenvalue of $A$.*

   *Furthermore, if $A$ is singular then every non-zero vector in $\mathcal{N}(A)$ is an eigenvector of $A$ with eigenvalue $0$.*

   **Remark.** When $\dim(\mathcal{N}(A)) \geq 2$, we do not expect any two arbitrary non-zero vectors in $\mathcal{N}(A)$ to be scalar multiples of each other. This result reminds us that we should not expect eigenvectors corresponding to the same eigenvalue of $A$ to be non-zero scalar multiples of each other.

10. **Lemma (2).**

    *Let $A$ be an $(n \times n)$-square matrix. Suppose $\lambda$ is a real number. Then the statements below hold:*

    (a) *$\lambda$ is an eigenvalue of $A$ if and only if $A - \lambda I_n$ is singular.*

    (b) *Now suppose $\lambda$ is an eigenvalue of $A$ indeed. Then for any non-zero vector $\mathbf{x}$ in $\mathbb{R}^n$, $\mathbf{x}$ is an eigenvector of $A$ with eigenvalue $\lambda$ if and only if $\mathbf{x} \in \mathcal{N}(A - \lambda I_n)$.*

11. **Definition. (Eigenspace.)**

    *Let $A$ be an $(n \times n)$-square matrix. Suppose $\lambda$ be an eigenvalue of $A$. Then $\mathcal{N}(A - \lambda I_n)$ is called the eigenspace of $A$ with eigenvalue $\lambda$. It is denoted by $\mathcal{E}_A(\lambda)$.*

    *The dimension of $\mathcal{E}_A(\lambda)$ is called the geometric multiplicity of the eigenvalue $\lambda$ of $A$.*

12. **Theorem (D).**

Let $A$ is an $(n \times n)$-square matrix. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_p$ are all the eigenvalues of $A$, pairwise distinct.

Then the statements below are logically equivalent:

(a) $A$ is diagonalizable.

(b) The sum of the respective geometric multiplicities of $\lambda_1, \lambda_2, \cdots, \lambda_p$, as eigenvalues of $A$, equals $n$.

Now suppose $A$ is indeed diagaonalizable.

For each $k = 1, 2, \cdots, p$, write $\dim(\mathcal{E}_A(\lambda_k)) = n_k$, and suppose that $\mathbf{v}_{k,1}, \mathbf{v}_{k,2}, \cdots, \mathbf{v}_{k,n_k}$ constitute a basis for $\mathcal{E}_A(\lambda_k)$. Then a basis for $\mathbb{R}^n$ is constituted by

$$\begin{aligned}
&\mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \cdots, \mathbf{v}_{1,n_1}, \\
&\mathbf{v}_{2,1}, \mathbf{v}_{2,2}, \cdots, \mathbf{v}_{2,n_2}, \\
&\quad\quad\quad \vdots \\
&\mathbf{v}_{p,1}, \mathbf{v}_{p,2}, \cdots, \mathbf{v}_{p,n_p}.
\end{aligned}$$

**Proof.** Omitted. (The argument is not difficult at a conceptual level; we can certainly give it within the context of this course. However it will be tedious unless we introduce the notion of *direct sum.*)

13. **Illustration of the content of Theorem (D).**

(a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of $A$ with respective eigenvalues $1, -2$.

The only eigenspaces of $A$ are $\mathcal{E}_A(1), \mathcal{E}_A(-2)$.

The dimension of $\mathcal{E}_A(1)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(-2)$ is 1, with a basis given by $\mathbf{u}_2$.

Since $\dim(\mathcal{E}_A(1)) + \dim(\mathcal{E}_A(-2)) = 2 = \dim(\mathbb{R}^2)$, $A$ is expected to be diagonalizable.

A diagonalization of $A$ is given by $U^{-1}AU = \operatorname{diag}(1, -2)$, in which $U = \begin{bmatrix} \mathbf{u}_1 & | & \mathbf{u}_2 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of $A$ with respective eigenvalues $1, 2, 3$.

The only eigenspaces of $A$ are $\mathcal{E}_A(1), \mathcal{E}_A(2), \mathcal{E}_A(3)$.

The dimension of $\mathcal{E}_A(1)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(2)$ is 1, with a basis given by $\mathbf{u}_2$.

The dimension of $\mathcal{E}_A(3)$ is 1, with a basis given by $\mathbf{u}_3$.

Since $\dim(\mathcal{E}_A(1)) + \dim(\mathcal{E}_A(2)) + \dim(\mathcal{E}_A(3)) = 3 = \dim(\mathbb{R}^3)$, $A$ is expected to be diagonalizable.

A diagonalization of $A$ is given by $U^{-1}AU = \operatorname{diag}(1, 2, 3)$, in which $U = \begin{bmatrix} \mathbf{u}_1 & | & \mathbf{u}_2 & | & \mathbf{u}_3 \end{bmatrix}$.

(c) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of $A$ with respective eigenvalues $4, 1, 1$.

The only eigenspaces of $A$ are $\mathcal{E}_A(4), \mathcal{E}_A(1)$.

The dimension of $\mathcal{E}_A(4)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(1)$ is 2, with a basis given by $\mathbf{u}_2, \mathbf{u}_3$.

Since $\dim(\mathcal{E}_A(4)) + \dim(\mathcal{E}_A(1)) = 3 = \dim(\mathbb{R}^3)$, $A$ is expected to be diagonalizable.

A diagonalization of $A$ is given by $U^{-1}AU = \operatorname{diag}(4, 1, 1)$, in which $U = \begin{bmatrix} \mathbf{u}_1 & | & \mathbf{u}_2 & | & \mathbf{u}_3 \end{bmatrix}$.

(d) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of $A$ with respective eigenvalues $1, -1, 3, -3$.

The only eigenspaces of $A$ are $\mathcal{E}_A(1), \mathcal{E}_A(-1), \mathcal{E}_A(3), \mathcal{E}_A(-3)$.

The dimension of $\mathcal{E}_A(1)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(-1)$ is 1, with a basis given by $\mathbf{u}_2, \mathbf{u}_3$.

The dimension of $\mathcal{E}_A(3)$ is 1, with a basis given by $\mathbf{u}_3$.

The dimension of $\mathcal{E}_A(-3)$ is 1, with a basis given by $\mathbf{u}_4, \mathbf{u}_3$.

Since $\dim(\mathcal{E}_A(1)) + \dim(\mathcal{E}_A(-1)) + \dim(\mathcal{E}_A(3)) + \dim(\mathcal{E}_A(-3)) = 4 = \dim(\mathbb{R}^4)$, $A$ is expected to be diagonalizable.

A diagonalization of $A$ is given by $U^{-1}AU = \operatorname{diag}(1, -1, 3, -3)$, in which $U = \begin{bmatrix} \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \end{bmatrix}$.

(e) Let $b$ be a real number, and $A = \begin{bmatrix} b & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$b$ is the only eigenvalue of $A$, and and every eigenvector of $A$ is a non-zero scalar multiple of $\mathbf{u}$.

The only eigenspace of $A$ is $\mathcal{E}_A(b)$, which is of dimension 1.

Therefore $A$ is not diagonalizable.

(f) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

$A$ has no eigenvalues, and hence no eigenspace.

Therefore $A$ is not diagonalizable.

14. **Theorem (3).**

Let $A$ be an $(n \times n)$-square matrix. Suppose $A$ is diagonalizable, with a diagonalization $U^{-1}AU = D$, for some non-singular $(n \times n)$-square matrix $U$ and for some $(n \times n)$-diagonal matrix $D$.

Then the statements below hold:

(a) For each positive integer $p$, $A^p$ is diagonalizable, with a diagonalization given by $U^{-1}A^pU = D^p$.

(b) Suppose $A$ is non-singular. Then $D$ is non-singular, and $A^{-1}$ is diagonalizable, with a diagonalization given by $U^{-1}A^{-1}U = D^{-1}$.

**Proof of Theorem (3).** Exercise.

**Remark.** This result tells us that when $A$ is diagonalizable, it will be easy to find its positive powers. (Why? Because it is easy to find the positive powers of a diagonal matrix.)

15. **Examples on application of Theorem (3).**

(a) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

It happens that $A$ is diagonalizable, with a diagonalization given by $U^{-1}AU = \operatorname{diag}(1, 2, 3)$, in which $U = \begin{bmatrix} \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

Then $A = U \operatorname{diag}(1, 2, 3)U^{-1}$.

Note that $U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix}$.

Then for each positive integer $p$, we have

$$
\begin{aligned}
A^p &= U(\operatorname{diag}(1, 2, 3))^p U^{-1} = U(\operatorname{diag}(1, 2^p, 3^p))U^{-1} \\
&= \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^p & 0 \\ 0 & 0 & 3^p \end{bmatrix} \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 + 2^p & 1/2 - 2 \cdot 2^p + (3/2) \cdot 3^p \\ 0 & 2^p & -2 \cdot 2^p + 2 \cdot 3^p \\ 0 & 0 & 3^p \end{bmatrix}.
\end{aligned}
$$

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(b) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

It happens that $A$ is diagonalizable, with a diagonalization given by $U^{-1}AU = \text{diag}(4,1,1)$, in which $U = $
$\begin{bmatrix} \mathbf{u}_1 & | & \mathbf{u}_2 & | & \mathbf{u}_3 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Then $A = U \,\text{diag}(4,1,1)U^{-1}$.

Note that $U^{-1} = \dfrac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

Then for each positive integer $p$, we have

$$
\begin{aligned}
A^p &= U(\text{diag}(4,1,1))^p U^{-1} = U(\text{diag}(4^p,1,1))U^{-1} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4^p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 4^p & 4^p - 3 & 4^p + 3 \\ 4^p - 1 & 4^p + 2 & 4^p - 1 \\ 4^p + 1 & 4^p + 1 & 4^p - 2 \end{bmatrix}.
\end{aligned}
$$