MATH1030 Rank-nullity Formula.

1. **Definition. (Nullity, column rank, row rank of a matrix.)**

   Let A be a $(p \times q)$-matrix.

   (a) *The nullity of A is defined to be the dimension of the null space of A. It is denoted by $n(A)$.*

   (b) *The column rank of A is defined to be the dimension of the column space of A. It is denoted by $r_{col}(A)$.*

   (c) *The row rank of A is defined to be the dimension of the row space of A. It is denoted by $r_{row}(A)$.*

2. **Theorem (K).**

   Let A be a $(p \times q)$-matrix.

   Denote by A' the reduced row-echelon form which is row-equivalent to A, and suppose the rank of A' is $r(A)$.

   Then the statements below hold:

   (a) $r(A) = r_{col}(A) = r_{row}(A)$.

   (b) $n(A) + r(A) = q$.

   (c) $r(A^t) = r(A)$, *and* $n(A^t) + r(A) = p$.

   Remarks.

   - The column space of A is a subspace of $\mathbb{R}^q$ while the row space of A is a subspace of $\mathbb{R}^p$. So despite the equality $r_{col}(A) = r_{row}(A)$, we do not expect these two objects to be 'comparable'. In fact, what is important is that despite their distinction as objects, their respective dimensions are the same.
   - The equality '$n(A) + r(A) = q$' is referred to as the 'Rank-nullity Formula' (for the matrix A with q columns).

3. **Corollary to Theorem (K).**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_t$ be vectors in $\mathbb{R}^q$. Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_t\ ]$.

   Then the dimension of $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_t\})$ *is $r(U)$.*

4. **Proof of Theorem (K).**

   (a) The number of vectors in a basis for $\mathcal{C}(A)$ is the same as the number of pivot columns in A', which is the rank of A'. Hence $r(A) = r_{col}(A)$.

   The number of vectors in a basis for $\mathcal{R}(A)$ is the number of non-zero rows in A', which is also the rank of A'. Hence $r(A) = r_{row}(A)$.

   (b) The nullity of A is the same as the number of free columns in A'.

   Then $n(A) = q - r(A)$.

   Therefore $n(A) + r(A) = q$.

   (c) Note that $\mathcal{C}(A^t) = \mathcal{R}(A)$.

   We have $r(A^t) = r_{col}(A^t) = r_{row}(A) = r(A)$.

   Then $n(A^t) + r(A) = n(A^t) + r(A^t) = p$.

5. **Illustrations of the content of Theorem (K).**

   (a) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 4 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$, and write $B = A^t$.

   The reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   By direct inspection on A', we see that $r(A) = 3$ and $n(A) = 1$.

   As expected from theory, we have $n(A) + r(A) = 4$.

   Note that $B = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 1 & 0 & 4 & 2 \\ 1 & -1 & 4 & 1 \\ 1 & 0 & 3 & 1 \end{bmatrix}$.

   The reduced row-echelon form B' which is row-equivalent to B is given by $B' = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Note that B' is not the same as the transpose of A'. However $r(B) = r(B') = 3$; so, as expected from theory, $r(B) = r(A)$.

By direct inspection on B', we see that $r(B) = 3$ and $n(B) = 1$.

As expected from theory, $n(B) + r(B) = 4$.

(b) Let $A = \begin{bmatrix} 1 & 2 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 & 5 \\ 2 & 6 & 5 & 9 & 6 \end{bmatrix}$, and write $B = A^t$.

The reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & -1 & 4 \end{bmatrix}$.

By direct inspection on A', we see that $r(A) = 3$ and $n(A) = 2$.

As expected from theory, we have $n(A) + r(A) = 5$.

Note that $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 2 & 3 & 5 \\ 3 & 4 & 9 \\ 4 & 5 & 6 \end{bmatrix}$.

The reduced row-echelon form B' which is row-equivalent to B is given by $B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note that B' is not the same as the transpose of A'. However $r(B) = r(B') = 3$; so, as expected from theory, $r(B) = r(A)$.

By direct inspection on B', we see that $r(B) = 3$ and $n(B) = 0$.

As expected from theory, $n(B) + r(B) = 3$.

(c) Let $A = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}$, and write $B = A^t$.

The reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

By direct inspection on A', we see that $r(A) = 3$ and $n(A) = 4$.

As expected from theory, we have $n(A) + r(A) = 7$.

Note that $B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 0 & -4 & -6 \\ -1 & 2 & -1 & -1 \\ 1 & 3 & 3 & 5 \\ 0 & 5 & 2 & 4 \\ 2 & -7 & 1 & 0 \\ 0 & 12 & 5 & 10 \end{bmatrix}$.

The reduced row-echelon form B' which is row-equivalent to B is given by $B' = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Note that B' is not the same as the transpose of A'. However $r(B) = r(B') = 3$; so, as expected from theory, $r(B) = r(A)$.

By direct inspection on B', we see that $r(B) = 3$ and $n(B) = 1$.

As expected from theory, $n(B) + r(B) = 4$.

6. **Theorem (1).**

Suppose A is a $(p \times q)$-matrix.

Then the inequalities below hold:

(a) $r(A) \leq p$.

(b) $r(A) \leq q$.

(c) $n(A) \geq q - p$.

Proof of Theorem (1). The first two inequalities follow immediately from the definition of $r(A)$ as the dimension of the column space of A and also as the dimension of the row space of A. As for the third, it is a consequence of the equality $n(A) = q - r(A)$.

7. **Lemma (2).**

   Suppose A is a $(p \times q)$-matrix, and B is an $(q \times s)$-matrix. Then $\mathcal{N}(B)$ is a subspace of $\mathcal{N}(AB)$.

   Proof of Lemma (2).

   Suppose A is a $(p \times q)$-matrix, and B is a $(q \times s)$-matrix.

   By definition, AB is an $(p \times s)$-matrix. Note that $\mathcal{N}(B), \mathcal{N}(AB)$ are both subspaces of $\mathbb{R}^s$.

   Pick any vector $\mathbf{v} \in \mathbb{R}^s$. Suppose $\mathbf{v} \in \mathcal{N}(B)$. Then by definition, $B\mathbf{v} = \mathbf{0}_q$.

   We have $(AB)\mathbf{v} = A(B\mathbf{v}) = A\mathbf{0}_q = \mathbf{0}_p$. Then by definition $\mathbf{v} \in \mathcal{N}(AB)$.

   It follows that $\mathcal{N}(B)$ is a subspace of $\mathcal{N}(AB)$.

8. **Theorem (3).**

   Suppose A is a $(p \times q)$-matrix, and B is an $(q \times s)$-matrix.

   Then the inequalities below hold:

   (a) $n(B) \leq n(AB)$.

   (b) $r(AB) \leq r(B)$.

   (c) $r(AB) \leq r(A)$.

   (d) $n(A) + s \leq n(AB) + q$.

9. **Proof of Theorem (3).**

   Suppose A is a $(p \times q)$-matrix, and B is a $(q \times s)$-matrix.

   (a) By Lemma (2), $\mathcal{N}(B)$ is a subspace of $\mathcal{N}(AB)$.
   Then $n(B) = \dim(\mathcal{N}(B)) \leq \dim(\mathcal{N}(AB)) = n(AB)$.

   (b) By the Rank-nullity Formula, we have $n(B) + r(B) = s$, and $n(AB) + r(AB) = s$.
   Then $r(AB) = s - n(AB) \leq s - n(B) = r(B)$.

   (c) Note that $B^t A^t = (AB)^t$.
   Then, also by Lemma (2), $\mathcal{N}(A^t)$ is a subspace of $\mathcal{N}((AB)^t)$.
   Therefore $n(A^t) = \dim(\mathcal{N}(A^t)) \leq \dim(\mathcal{N}((AB)^t)) = n((AB)^t)$
   By the Rank-nullity Formula, we have $n(A^t) + r(A^t) = p$ and $n((AB)^t) + r((AB)^t) = p$.
   Then $r(AB) = r((AB)^t) = p - n((AB)^t) \leq p - n(A^t) = r(A^t) = r(A)$.

   (d) Again by the Rank-nullity Formula, we have $n(A) + r(A) = n$ and $n(AB) + r(AB) = s$.
   Then $s - n(AB) = r(AB) \leq r(A) = q - n(A)$.
   Therefore $n(A) + s \leq n(AB) + q$.