MATH1030 Inequalities on dimension.

1. **Definition. (Subspaces of subspace of $\mathbb{R}^n$.)**

   Let V, W be subspaces of $\mathbb{R}^n$.

   We say V is a subspace of W if and only if the statement (†) holds:

   (†) *For any $\mathbf{x} \in \mathbb{R}^n$, if $\mathbf{x} \in V$ then $\mathbf{x} \in W$.*

   Remark. In plain words, the statement (†) reads: '*every vector of V belongs to W*'.

2. Again recall the Replacement Theorem (Theorem (F)) in the handout *More on minimal spanning set*:

   Let W be a subspace of $\mathbb{R}^n$.
   Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in W. Suppose none of these vectors is the zero vector.
   Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ are linearly independent.
   Further suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for W.
   Then, $q \geq p$, and there is a basis for W which is constituted by $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ together with some $q-p$ vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$.

   This result, combined with the notion of *subspaces of a subspace of* $\mathbb{R}^n$, lead to Theorem (I), which is a useful tool for comparing various subspaces of $\mathbb{R}^n$.

3. **Theorem (I).**

   Let V, W be subspaces of $\mathbb{R}^n$.

   Suppose V is a subspace of W.

   Then $\dim(V) \leq \dim(W)$. Equality holds if and only if $V = W$.

   Remark. From the handout *Dimension*, we have learnt that

   - every subspace of $\mathbb{R}^n$ is of dimension at most n, and
   - $\mathbb{R}^n$ is the one and only one n-dimensional subspace of $\mathbb{R}^n$.

   What we have learnt earlier can be seen as a manifestation of Theorem (I) in the special case in which $W = \mathbb{R}^n$.

4. **Proof of Theorem (I).**

   Let V, W be subspaces of $\mathbb{R}^n$.

   Suppose V is a subspace of W.

   Write $\dim(V) = k$. Pick some basis for V, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$. Then by assumption, each of them belongs to W.

   (a) By definition, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent.
   So they are k linearly independent vectors in W.
   Hence $\dim(V) = k \leq \dim(W)$ by Theorem (H).

   (b) Suppose $V = W$. Then $\dim(V) = \dim(W)$.

   (c) Suppose $\dim(V) = \dim(W)$. Then $\dim(W) = k$.
   Pick some basis for W, say, $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$.
   Recall that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are k linearly independent vectors in W.
   Then by the Replacement Theorem, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, together with possibly some vectors from amongst $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$, constitute a basis for W.
   However, since $\dim(W) = k$, the k vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ already constitute a basis for W.
   It follows that $W = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}) = V$.

5. **Corollary (1) to Theorem (I).**

   Let V, W be subspaces of $\mathbb{R}^n$. Suppose $\dim(W) = p$.

   Further suppose V is a subspace of W.

   Also suppose that there are p vectors in V which are linearly independent.

   Then $V = W$.

6. **Proof of Corollary (1) to Theorem (I).**

   Let V, W be subspaces of $\mathbb{R}^n$. Suppose $\dim(W) = p$.

   Further suppose V is a subspace of W.

   Also suppose that there are p vectors in V which are linearly independent.

- Since V is a subset of W, we have $\dim(V) \leq \dim(W)$.
- Since there are p vectors in V which are linearly independent, we have $\dim(V) \geq p = \dim(W)$.

Then $\dim(V) = \dim(W)$. Now, by Theorem (I), since V is a subspace of W, we have $V = W$.

7. **Corollary (2) to Theorem (I).**

Let V, W be subspaces of $\mathbb{R}^n$. Suppose $\dim(V) = p$.

Further suppose V is a subspace of W.

Also suppose that there are p vectors of W so that every vector of W is a linear combination of these p vectors.

Then $V = W$.

8. **Proof of Corollary (2) to Theorem (I).**

Let V, W be subspaces of $\mathbb{R}^n$. Suppose $\dim(V) = p$.

Further suppose V is a subspace of W.

Also suppose that there are p vectors of W, say, $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$, so that every vector of W is a linear combination of these p vectors.

- Since $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ are all vectors in W, every linear combination of $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ is a vector in W.
  Then $W = \mathsf{Span}\ \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p\}$.
  Therefore there is a basis for W from amongst the p vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$. Hence $\dim(W) \leq p = \dim(V)$.
- Since V is a subspace of W, we have $\dim(V) \leq \dim(W)$.

Then $\dim(V) = \dim(W)$. Now, by Theorem (I), since V is a subset of W, we have $V = W$.

9. **Theorem (J). (Re-formulation of the notion of basis in terms of dimension.)**

Let W be a subspace of $\mathbb{R}^n$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in W.

Then the statements below are logically equivalent:

($\sharp$) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *constitute a basis for W.*

($\natural$) $\dim(W) = p$, *and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.*

($\flat$) $\dim(W) = p$, *and every vector of W is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.*

10. **Proof of Theorem (J).**

Let W be a subspace of $\mathbb{R}^n$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in W.

[The statement ($\sharp$) implies each of the statements ($\natural$), ($\flat$) immediately.
What matters is whether whether each of the statements ($\natural$), ($\flat$) separately implies ($\sharp$).]

- [We ask whether ($\natural$) implies ($\sharp$).]
  Suppose $\dim(W) = p$ and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.
  Define $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\})$. [We want to show that $V = W$.]
  By definition, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for the p-dimensional subspace V of $\mathbb{R}^n$.
  Now we have $\dim(V) = p = \dim(W)$.
  Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are vectors in W, every linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ belongs to W.
  Then V is a subspace of W. Now, by Corollary (2) to Theorem (I), we have $V = W$.
  Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for W.
- [We ask whether ($\flat$) implies ($\sharp$).]
  Suppose $\dim(W) = p$, and every vector in W is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.
  Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are vectors in W, every linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ is a vector in W.
  Then there is a basis for W, with, say, q vectors, from amongst the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$. Without loss of generality, assume they are $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$. These q vectors are linearly independent.
  Since these q vectors constitute a basis for W, we have $\dim(W) = q$.
  Then $p = \dim(W) = q$. Therefore $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.
  Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for W.

11. **Theorem (J'). (Re-formulation of Theorem (J) in terms of systems of equations.)**

Let W be a subspace of $\mathbb{R}^n$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in W, and U is the $(n \times p)$-matrix given by $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_p\ ]$.

Then the statements below are logically equivalent:

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *is a basis for W.*

(b) $\dim(W) = p$, *and the homogeneous system $\mathcal{LS}(U, \mathbf{0})$ has no non-trivial solution.*

(c) $\dim(W) = p$, *and for any $\mathbf{b} \in V$, the system $\mathcal{LS}(U, \mathbf{b})$ is consistent.*