MATH1030 Duality between spanning and linear independence.

1. **Lemma (ζ).**

   Let $\mathbf{y} \in \mathbb{R}^p$. The statements below are logically equivalent:

   (a) $\mathbf{y} = \mathbf{0}_p$.

   (b) *For any* $\mathbf{z} \in \mathbb{R}^p$, $\mathbf{y}^t\mathbf{z} = 0$.

2. **Proof of Lemma (ζ).**

   Let $\mathbf{y} \in \mathbb{R}^p$. The statements below are logically equivalent:

   - Suppose $\mathbf{y} = \mathbf{0}_p$.
     Pick any $\mathbf{z} \in \mathbb{R}^p$. We have $\mathbf{y}^t\mathbf{z} = \mathbf{0}_p{}^t\mathbf{z} = 0$.
   - Suppose that for any $\mathbf{z} \in \mathbb{R}^p$, $\mathbf{y}^t\mathbf{z} = 0$.
     Denote the j-th entry of $\mathbf{y}$ by y_j for each $j = 1, 2, \cdots, p$.
     We have $\mathbf{y} = y_1\mathbf{e}_1^{(p)} + y_2\mathbf{e}_2^{(p)} + \cdots + y_p\mathbf{e}_p^{(p)}$.
     Then, for each $j = 1, 2, \cdots, p$, we have $y_j = \mathbf{y}^t\mathbf{e}_j^{(p)} = 0$.
     Therefore $\mathbf{y} = \mathbf{0}_p$.

3. **Theorem (η).**

   Let A be an $(m \times n)$-matrix.

   Suppose $\mathcal{C}(A) = \mathbb{R}^n$. Then $\mathcal{N}(A^t) = \{\mathbf{0}_m\}$.

4. **Proof of Theorem (η).**

   Let A be an $(m \times n)$-matrix.

   Suppose $\mathcal{C}(A) = \mathbb{R}^n$.

   [We want to verify that $\mathbf{0}_m$ is the only vector in $\mathcal{N}(A^t)$.]

   Pick any $\mathbf{v} \in \mathbb{R}^m$. Suppose $\mathbf{v} \in \mathcal{N}(A^t)$. [Ask: is it true that $\mathbf{v} = \mathbf{0}_m$?]

   We verify that for any $\mathbf{w} \in \mathbb{R}^m, \mathbf{v}^t\mathbf{w} = 0$:

   - Pick any $\mathbf{w} \in \mathbb{R}^m$.
     Since $\mathcal{C}(A) = \mathbb{R}^m$, there exists some $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{w} = A\mathbf{x}$.
     Then $\mathbf{v}^t\mathbf{w} = \mathbf{v}^t(A\mathbf{x}) = (\mathbf{v}^tA)\mathbf{x} = (A^t\mathbf{v})^t\mathbf{x}$.
     Recall that $\mathbf{v} \in \mathcal{N}(A^t)$. Then by definition, $A^t\mathbf{v} = \mathbf{0}_n$.
     Now we have $\mathbf{v}^t\mathbf{w} = (A^t\mathbf{v})^t\mathbf{x} = \mathbf{0}_n{}^t\mathbf{x} = 0$.
     Therefore, by Lemma (ζ), $\mathbf{v} = \mathbf{0}_m$.

   It follows that $\mathcal{N}(A^t) = \{\mathbf{0}_m\}$.

5. The converse of Theorem (η) is also true. The argument relies on the result below:

   Theorem (θ).

   Let C be an $(p \times q)$-matrix. Suppose K is a non-singular $(p \times p)$-square matrix. Then the equalities below hold:

   (a) $\mathcal{N}(KC) = \mathcal{N}(C)$.

   (b) $\mathcal{R}(KC) = \mathcal{R}(C)$.

6. **Proof of Theorem (θ).**

   Let C be an $(p \times q)$-matrix. Suppose K is a non-singular $(p \times p)$-square matrix.

   (a) $\mathcal{N}(KC)$ is the solution set of the homogeneous system $\mathcal{LS}(KC, \mathbf{0}_p)$.
   $\mathcal{N}(C)$ is the solution set of the homogeneous system $\mathcal{LS}(C, \mathbf{0}_p)$.
   By assumption, KC is row-equivalent to C. Then $\mathcal{LS}(C, \mathbf{0}_p)$ is equivalent to $\mathcal{LS}(KC, \mathbf{0}_p)$. (So every solution of the former is a solution of the latter, and every solution of the latter is a solution of the former.)
   Therefore $\mathcal{N}(KC) = \mathcal{N}(C)$.

   (b) According to Theorem (δ) in the handout *Transpose and row space*, we have $\mathcal{R}(KC) = \mathcal{R}(K)$.

7. **Theorem (ι). (Converse of Theorem (η).)**

   Let A be an $(m \times n)$-matrix.

   Suppose $\mathcal{N}(A^t) = \{\mathbf{0}_m\}$. Then $\mathcal{C}(A) = \mathbb{R}^n$.

   Proof of Theorem (ι).

   Let A be an $(m \times n)$-matrix.

   Suppose $\mathcal{N}(A^t) = \{\mathbf{0}_m\}$.

   Denote by the B the reduced row-echelon form which is row-equivalent to A^t.

   Then $\mathcal{N}(B) = \mathcal{N}(A^t)$. (Why?)

   Since B is a reduced row-echelon form and $\mathcal{N}(B) = \{\mathbf{0}_m\}$, every column of B is a pivot column.

   Then $n \geq m$, and $B = \left[\begin{array}{c} I_m \\ \hline \mathcal{O}_{(n-m)\times m} \end{array}\right]$.

   Therefore $B^t = [\ I_m \mid \mathcal{O}_{m\times(n-m)}\ ]$. We have $\mathcal{C}(B^t) = \mathbb{R}^m$.

   Recall that by Theorem (ϵ), we have $\mathcal{R}(A^t) = \mathcal{R}(B)$.

   Then $\mathcal{C}(A) = \mathcal{R}(A^t) = \mathcal{R}(B) = \mathcal{C}(B^t) = \mathbb{R}^m$.

8. Recall that we say some given vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ spans $\mathbb{R}^p$ exactly when every vector in $\mathbb{R}^p$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$. In set language, we may present this as in the form of the equality $\mathbb{R}^p = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\})$.

   Combining Theorem (η) and Theorem (ι), we obtain the result below:

   Theorem (κ). (Duality between spanning and linear independence.)

   Let A be an $(m \times n)$-matrix.

   (a) *The statements below are logically equivalent:*
      i. *The columns of A (regarded as column vectors in $\mathbb{R}^m$) span $\mathbb{R}^m$.*
      ii. $\mathcal{C}(A) = \mathbb{R}^m$.
      iii. $\mathcal{N}(A^t) = \{\mathbf{0}_m\}$.
      iv. *The columns of A^t (regarded as column vectors in $\mathbb{R}^n$) are linearly independent.*

   (b) *The statements below are logically equivalent:*
      i. *The columns of A^t (regarded as column vectors in $\mathbb{R}^n$) span $\mathbb{R}^n$.*
      ii. $\mathcal{C}(A^t) = \mathbb{R}^n$.
      iii. $\mathcal{N}(A) = \{\mathbf{0}_n\}$.
      iv. *The columns of A (regarded as column vectors in $\mathbb{R}^m$) are linearly independent.*

9. Theorem (κ) can also be re-formulated in terms of systems of linear equations. Such a formulation is useful in various branches of 'applied mathematics'.

   Theorem (λ). (Re-formulation of Theorem (κ).)

   Let A be an $(m \times n)$-matrix.

   (a) *The statements below are logically equivalent:*
      i. *For any $\mathbf{b} \in \mathbb{R}^m$, the system $\mathcal{LS}(A, \mathbf{b})$ is consistent.*
      ii. $\mathcal{C}(A) = \mathbb{R}^m$.
      iii. $\mathcal{N}(A^t) = \{\mathbf{0}_m\}$.
      iv. *The trivial solution is the only solution of the system $\mathcal{LS}(A^t, \mathbf{0}_n)$.*

   (b) *The statements below are logically equivalent:*
      i. *For any $\mathbf{c} \in \mathbb{R}^n$, the system $\mathcal{LS}(A^t, \mathbf{c})$ is consistent.*
      ii. $\mathcal{C}(A^t) = \mathbb{R}^n$.
      iii. $\mathcal{N}(A) = \{\mathbf{0}_n\}$.
      iv. *The trivial solution is the only solution of the system $\mathcal{LS}(A, \mathbf{0}_m)$.*