1. Recall the definition for the notion *transpose of a matrix* from the handout *Miscellanies on matrices*:

   Let A be an $(m \times n)$-matrix, whose (i, j)-th entry is denoted by a_{ij}.

   The $(n \times m)$-matrix whose (k, ℓ)-th entry is given by $a_{\ell k}$ is called the transpose of A, and is denoted by A^t.

   $$\text{(So } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \text{ whereas } A^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{13} & a_{23} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.\text{)}$$

2. **Theorem (α). (Basic properties of transpose.)**

   The statements below hold:

   (a) *Suppose A, B are $(m \times n)$-matrices. Then $(A + B)^t = A^t + B^t$.*

   (b) *Suppose A is an $(m \times n)$-matrix, and α is a real number. Then $(\alpha A)^t = \alpha A^t$.*

   (c) *Suppose A is an $(m \times n)$-matrix, and B is an $(n \times p)$-matrix. Then $(AB)^t = B^t A^t$.*

   (d) *Suppose A is an $(m \times n)$-matrix. Then $(A^t)^t = A$.*

   Proof of Theorem (α). Exercise. (It is necessary to go back to the definition for equalities between matrices in terms of equalities between respective entries.)

3. **Theorem (β). (Transpose and nonsingularity.)**

   Let A be an $(n \times n)$-square matrix.

   Suppose A is non-singular and invertible.

   Then A^t is non-singular and invertible, and the matrix inverse of A^t is given by $(A^t)^{-1} = (A^{-1})^t$.

4. **Proof of Theorem (β).**

   Let A be an $(n \times n)$-square matrix. Suppose A is non-singular and invertible.

   By assumption, the matrix inverse of A is well-defined. Write $B = A^{-1}$.

   By definition, $BA = I_n$ and $AB = I_n$.

   Then $B^t A^t = (AB)^t = {I_n}^t = I_n$.

   Also, $A^t B^t = (BA)^t = {I_n}^t = I_n$.

   Therefore, by definition, A^t is non-singular and invertible, and the matrix inverse of A^t is given by $(A^t)^{-1} = B^t = (A^{-1})^t$.

5. **Definition. (Row space of a matrix.)**

   Let G be an $(m \times n)$-matrix.

   The row space of G is defined to be the column space of the $(n \times m)$-matrix G^t. It is denoted by $\mathcal{R}(G)$.

   Remark. Denote the rows of G, from top to bottom, by $\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_m$. (So $G = \begin{bmatrix} \mathbf{g}_1 \\ \hline \mathbf{g}_2 \\ \hline \vdots \\ \hline \mathbf{g}_m \end{bmatrix}$.)

   Then, according to the 'dictionary' between the notions of *span* and *column space*, we have $\mathcal{R}(G) = \mathcal{C}(G^t) = \mathsf{Span}\left(\{{\mathbf{g}_1}^t, {\mathbf{g}_2}^t, \cdots, {\mathbf{g}_m}^t\}\right)$.

6. **Lemma (γ).**

   Suppose H is an $(n \times p)$-matrix, and B is a non-singular $(p \times p)$-matrix. Then $\mathcal{C}(HB) = \mathcal{C}(H)$.

   Remark. The conclusion in Lemma (γ) is a set equality, which reads:

   Both (†) and (‡) below hold:

   (†) *For any $\mathbf{y} \in \mathbb{R}^p$, if $\mathbf{y} \in \mathcal{C}(HB)$ then $\mathbf{y} \in \mathcal{C}(H)$.*

   (‡) *For any $\mathbf{z} \in \mathbb{R}^p$, if $\mathbf{z} \in \mathcal{C}(H)$ then $\mathbf{z} \in \mathcal{C}(HB)$.*

7. **Proof of Lemma (γ).**

   Suppose H is an $(n \times p)$-matrix, and B is a non-singular $(p \times p)$-matrix.

   - Pick any $\mathbf{y} \in \mathbb{R}^n$. Suppose $\mathbf{y} \in \mathcal{C}(HB)$.
     By definition, there exists some $\mathbf{s} \in \mathbb{R}^p$ such that $\mathbf{y} = (HB)\mathbf{s}$.

     [We want to verify $\mathbf{y} \in \mathcal{C}(H)$.
     We are in fact trying to verify that there is some $\mathbf{u} \in \mathbb{R}^p$ for which the equality $\mathbf{y} = H\mathbf{u}$ holds.
     Ask: Can we name such a vector $\mathbf{u}$? How about naming $\mathbf{u}$ as $B\mathbf{s}$?]

     Take $\mathbf{u} = B\mathbf{s}$. By definition, $\mathbf{u} \in \mathbb{R}^p$.
     Also, $\mathbf{y} = (HB)\mathbf{s} = H(B\mathbf{s}) = H\mathbf{u}$.
     Then, by definition, $\mathbf{y} \in \mathcal{C}(H)$.
   - Pick any $\mathbf{z} \in \mathbb{R}^n$. Suppose $\mathbf{z} \in \mathcal{C}(H)$.
     By definition, there exists some $\mathbf{t} \in \mathbb{R}^p$ such that $\mathbf{z} = H\mathbf{t}$.

     [We want to verify $\mathbf{z} \in \mathcal{C}(HB)$.
     We are in fact trying to verify that there is some $\mathbf{v} \in \mathbb{R}^p$ for which the equality $\mathbf{z} = (HB)\mathbf{v}$ holds.
     Ask: Can we name such a vector $\mathbf{v}$? How about naming $\mathbf{v}$ as $B^{-1}\mathbf{t}$?]

     Take $\mathbf{v} = B^{-1}\mathbf{t}$. By definition, $\mathbf{v} \in \mathbb{R}^p$.
     Also, $\mathbf{z} = H\mathbf{t} = H(I_p\mathbf{t}) = H[(BB^{-1})\mathbf{t}] = H[B(B^{-1}t)] = H(B\mathbf{v}) = (HB)\mathbf{v}$.
     Then, by definition, $\mathbf{z} \in \mathcal{C}(HB)$.

   It follows that $\mathcal{C}(H) = \mathcal{C}(HB)$.

8. **Theorem (δ).**

   Suppose G is an $(m \times n)$-matrix, and A is a non-singular $(m \times m)$-matrix. Then $\mathcal{R}(AG) = \mathcal{R}(G)$.

   Proof of Theorem (δ).

   Suppose G is an $(m \times n)$-matrix, and A is a non-singular $(m \times m)$-matrix.

   Note that A^t is a non-singular $(m \times m)$-matrix.

   Then $\mathcal{R}(AG) = \mathcal{C}((AG)^t) = \mathcal{C}(G^tA^t) = \mathcal{C}(G^t) = \mathcal{R}(G)$.

   Remark. In plain words, this result is saying that

   > *the row space of a matrix is preserved upon multiplication of a non-singular square matrix from the left to matrix.*

   When we think in terms of row operations, this result is saying that

   > *the row space of a matrix is preserved upon the application of row operations on the matrix.*

9. **Theorem (ε).**

   Suppose G is an $(m \times n)$-matrix, and $\hat{G}$ is the reduced row-echelon form which is row-equivalent to G. Then the statements below hold:

   (a) $\mathcal{R}\left(\hat{G}\right) = \mathcal{R}(G)$.

   (b) *Denote the rank of $\hat{G}$ by r. Suppose $r > 0$.*
   Denote the top r rows of $\hat{G}$ by $\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \cdots, \hat{\mathbf{g}}_r$.
   Then $\hat{\mathbf{g}}_1^t, \hat{\mathbf{g}}_2^t, \cdots, \hat{\mathbf{g}}_r^t$ constitute a basis for $\mathcal{R}(G)$.

10. **Proof of Theorem (ε).**

    Suppose G is an $(m \times n)$-matrix, and $\hat{G}$ is the reduced row-echelon form which is row-equivalent to G.

    (a) There exists some non-singular $(m \times m)$-square matrix A such that $\hat{G} = AG$.
    Then $\mathcal{R}\left(\hat{G}\right) = \mathcal{R}(AG) = \mathcal{R}(G)$.

    (b) Denote the rank of $\hat{G}$ by r. Suppose $r > 0$.
    Denote the top r rows of $\hat{G}$ by $\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \cdots, \hat{\mathbf{g}}_r$.
    Note that the bottom $m - r$ rows of $\hat{G}$ are rows of zeros. Their respective transposes are the zero vector in $\mathbb{R}^n$.
    We verify that $\hat{\mathbf{g}}_1^t, \hat{\mathbf{g}}_2^t, \cdots, \hat{\mathbf{g}}_r^t$ constitute a basis for $\mathcal{R}\left(\hat{G}\right)$:

- We have $\mathcal{R}\left(\hat{G}\right) = \mathcal{C}\left(\hat{G}^t\right) = \mathsf{Span}\ (\{\hat{\mathbf{g}}_1^t, \hat{\mathbf{g}}_2^t, \cdots, \hat{\mathbf{g}}_r^t, \underbrace{\mathbf{0}_n, \mathbf{0}_n, \cdots, \mathbf{0}_n}_{m-r \text{ copies}}\}) = \mathsf{Span}\ (\{\hat{\mathbf{g}}_1^t, \hat{\mathbf{g}}_2^t, \cdots, \hat{\mathbf{g}}_r^t\})$.
- [We want to verify that $\hat{\mathbf{g}}_1^t, \hat{\mathbf{g}}_2^t, \cdots, \hat{\mathbf{g}}_r^t$ are linearly independent.]
  Label the pivot columns of $\hat{G}$, from left to right, by $d_1, d_2, \cdots, d_r$.
  Then by definition, for each $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, r$, the j-th entry c_{ij} of $\hat{\mathbf{g}}_i^t$ is given by

$$c_{ij} = \left\{ \begin{array}{lll} 1 & \text{if} & i = j \\ 0 & \text{if} & i \neq j \end{array} \right.$$

  Pick any $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathbb{R}$. Suppose $\alpha_1 \hat{\mathbf{g}}_\mathbf{1}^t + \alpha_2 \hat{\mathbf{g}}_\mathbf{2}^t + \cdots + \alpha_r \hat{\mathbf{g}}_\mathbf{r}^t = \mathbf{0}_n$.
  For each $j = 1, 2, \cdots, r$, the j-th entry of the vector $\alpha_1 \hat{\mathbf{g}}_\mathbf{1}^t + \alpha_2 \hat{\mathbf{g}}_\mathbf{2}^t + \cdots + \alpha_r \hat{\mathbf{g}}_\mathbf{r}^t$ is given by $\alpha_1 c_{1j} + \alpha_2 c_{2j} + \cdots + \alpha_r c_{rj} = \alpha_j$.
  The j-th entry of $\mathbf{0}_n$ is 0.
  Then $\alpha_j = 0$.
  Hence $\hat{\mathbf{g}}_1^t, \hat{\mathbf{g}}_2^t, \cdots, \hat{\mathbf{g}}_r^t$ are linearly independent.

It follows that $\hat{\mathbf{g}}_1^t, \hat{\mathbf{g}}_2^t, \cdots, \hat{\mathbf{g}}_r^t$ constitute a basis for $\mathcal{R}\left(\hat{G}\right)$. Hence they also constitute a basis for $\mathcal{R}(G)$.

11. Theorem (ε) suggests another method for determining a basis for the span of several vectors (which is different from the method described in the handout *Minimal spanning set*).

    'Algorithm' associated with Theorem (ε).

    Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be non-zero vectors in $\mathbb{R}^n$.

    We proceed to determine a basis for $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\})$ as described below:

    - **Step (1).**

      Form the $(p \times n)$-matrix $G = \left[\begin{array}{c} {\mathbf{u}_1}^t \\ \hline {\mathbf{u}_2}^t \\ \hline \vdots \\ \hline {\mathbf{u}_p}^t \end{array}\right]$.
    - **Step (2).**

      Obtain the reduced row-echelon form $\hat{G}$ which is row equivalent to G.
    - **Step (3).**

      Denote the rank of $\hat{G}$ by r.

      (Since G is not the zero matrix, $\hat{G}$ is not the zero matrix. The rank of $\hat{G}$ will be at least 1.)

      Denote the top r rows of $\hat{G}$ by $\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \cdots, \hat{\mathbf{g}}_r$.

      $\hat{\mathbf{g}}_1^t, \hat{\mathbf{g}}_2^t, \cdots, \hat{\mathbf{g}}_r^t$ constitute a basis for $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\})$.

12. **Illustrations.**

    (a) *Let* $\mathbf{u}_1 = \begin{bmatrix} 7 \\ 6 \\ 12 \\ 33 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 5 \\ 7 \\ 24 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 5 \end{bmatrix}$, *and* $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$.

    We want to obtain a basis for V.

    Define $G = \left[\begin{array}{c} {\mathbf{u}_1}^t \\ \hline {\mathbf{u}_2}^t \\ \hline {\mathbf{u}_3}^t \end{array}\right]$.

    We find the reduced row-echelon form $\hat{G}$ which is row equivalent to G:

$$G = \left[\begin{array}{cccc} 7 & 6 & 12 & 33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \end{array}\right] \longrightarrow \cdots\cdots\cdots \longrightarrow \hat{G} = \left[\begin{array}{cccc} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

    The rank of $\hat{G}$ is 3. For each i, denote the transpose of the i-th row of $\hat{G}$ by $\mathbf{t}_i$.

    We have $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

    A basis for V is constituted by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 7 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 2 \\ 5 \\ -1 \\ 9 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ -1 \\ -5 \\ 2 \\ 0 \end{bmatrix}$ *and* $V = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.

We want to obtain a basis for V*.*

Define $G = \begin{bmatrix} \mathbf{u}_1{}^t \\ \hline \mathbf{u}_2{}^t \\ \hline \mathbf{u}_3{}^t \\ \hline \mathbf{u}_4{}^t \end{bmatrix}$.

We find the reduced row-echelon form $\hat{G}$ which is row equivalent to G:

$$G = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow \hat{G} = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of $\hat{G}$ is 3. For each i, denote the transpose of the i-th row of $\hat{G}$ by $\mathbf{t}_i$.

We have $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$.

A basis for V is constituted by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \\ 5 \\ -7 \\ 12 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ -1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ -4 \\ -1 \\ 3 \\ 2 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ -6 \\ -1 \\ 5 \\ 4 \\ 0 \\ 10 \end{bmatrix}$ *and* $V = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.

We want to obtain a basis for V*.*

Define $G = \begin{bmatrix} \mathbf{u}_1{}^t \\ \hline \mathbf{u}_2{}^t \\ \hline \mathbf{u}_3{}^t \\ \hline \mathbf{u}_4{}^t \end{bmatrix}$.

We find the reduced row-echelon form $\hat{G}$ which is row equivalent to G:

$$G = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow \hat{G} = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of $\hat{G}$ is 3. For each i, denote the transpose of the i-th row of $\hat{G}$ by $\mathbf{t}_i$.

We have $\mathbf{t}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$.

A basis for V is constituted by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$.