MATH1030 How to obtain a basis for the intersection of two subspaces of $\mathbb{R}^n$.

1. In the handout *More on minimal spanning sets*, we learnt how to obtain a basis for the sum of two given subspaces of $\mathbb{R}^n$ whose bases are provided already.

   Here we find out how to obtain a basis for the intersection of two given subspaces of $\mathbb{R}^n$ whose bases are provided already.

2. Recall the definition for the notion of *intersection of sets of vectors in* $\mathbb{R}^n$:

   > *Let S, T be sets of vectors in $\mathbb{R}^n$.*
   > *The intersection of S, T is defined to be the set $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in S$ and $\mathbf{x} \in T\}$, and is denoted by $S \cap T$.*

3. **Lemma (1).**

   Suppose V, W are subspaces of $\mathbb{R}^n$. Then $V \cap W$ is a subspace of $\mathbb{R}^n$.

   Proof of Lemma (1). Exercise.

4. **Lemma (2).**

   Let D_1 be an $(m_1 \times n)$-matrix, and D_2 be an $(m_2 \times n)$-matrix. Suppose D is the $((m_1 + m_2) \times n)$-matrix given by $D = \left[\begin{array}{c} D_1 \\ \hline D_2 \end{array}\right]$.

   Then $\mathcal{N}(D) = \mathcal{N}(D_1) \cap \mathcal{N}(D_2)$.

   Proof of Lemma (2). Exercise. (This result is a special of a result stated in the handout *Geometry of solution sets for systems of linear equations*.)

5. Lemma (2) suggest how we can obtain a basis for the intersection of two given subspaces of $\mathbb{R}^n$, each of them being the null space of some matrices with the same number of columns.

   'Algorithm' for determining a basis for the intersection of the null spaces of two given matrices.

   Suppose B is an $(m_1 \times n)$-matrix, and C is an $(m_2 \times n)$-matrix. Suppose $V = \mathcal{N}(B)$ and $W = \mathcal{N}(C)$.

   Then we may proceed to determine a basis for $V \cap W$ as described in the 'algorithm' below:

   - **Step (1).**
     Form the matrix $A = \left[\begin{array}{c} B \\ \hline C \end{array}\right]$.
   - **Step (2).**
     Obtain the reduced row-echelon form A' which is row-equivalent to A.
     Denote the rank of A by r.
     If $r = n$ then $\mathcal{N}(A) = \{\mathbf{0}_n\}$.
     If $r < n$, proceed to Step (3).
   - **Step (3).**
     Suppose $r < n$. Write $p = n - r$.
     'Read off' from A' those p solutions, denoted by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$, of the system $\mathcal{LS}(A', \mathbf{0})$, for which exactly one of the free variables takes the value 1 and all other free variables take the value 0.
     These p vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for $\mathcal{N}(A)$. (This is guaranteed by Theorem (D) in the handout *Gaussian elimination and basis for null space*.)

   Remark. This is Theorem (D), proved in the handout *Gaussian elimination and basis for null space*:

   > *Let A be an $(m \times n)$-matrix, and A' be the reduced row-echelon form which is row-equivalent to A.*
   > *Suppose the rank of A' is r. Label the pivot columns of A', from left to right, by $d_1, d_2, \cdots, d_r$.*
   > *Write $p = n - r$. Suppose $p > 0$. Label the free columns of A', from left to right, by $f_1, f_2, \cdots, f_p$.*
   > *For each $h = 1, 2, \cdots, r$, and each $k = 1, 2, \cdots, p$, denote by s_{hk} the (d_h, f_k)-th entry of A'.*
   > *For each $k = 1, 2, \cdots, p$, define $\mathbf{u}_k$ to be the vector in $\mathbb{R}^n$ whose f_k-th entry is 1, whose f_j-th entry is 0 whenever $k \neq j$, and whose d_h-th entry is $-s_{hk}$ for each $h = 1, 2, \cdots, r$.*
   > *Then the statements below hold:*
   > (a) $\mathbf{u}_k \in \mathcal{N}(A)$ *for each $k = 1, 2, \cdots, p$.*
   > (b) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *are linearly independent.*
   > (c) *Every vector in $\mathcal{N}(A)$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.*

(d) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *constitute a basis for* $\mathcal{N}(A)$.

6. **Illustrations on how to determine a basis for the intersection of the null spaces of two given matrices.**

(a) *Let* $B = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \end{bmatrix}$, *and* $C = [\ 2 \quad 6 \quad 5 \quad 6\ ]$.

We want to determine a basis for $\mathcal{N}(B) \cap \mathcal{N}(C)$.

Define $A = \left[\frac{B}{C}\right]$. Then $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$, and $\mathcal{N}(A) = \mathcal{N}(B) \cap \mathcal{N}(C)$.

We obtain the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} = A'$$

Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\begin{cases} x_1 & & & + & 2x_4 & = & 0 \\ & x_2 & & - & 3x_4 & = & 0 \\ & & x_3 & + & 4x_4 & = & 0 \end{cases}$$

A basis for $\mathcal{N}(A)$ (which is $\mathcal{N}(B) \cap \mathcal{N}(C)$) is constituted by the vector $\mathbf{u}$, in which $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$.

(b) *Let* $B = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \end{bmatrix}$, *and* $C = \begin{bmatrix} 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix}$.

We want to determine a basis for $\mathcal{N}(B) \cap \mathcal{N}(C)$.

Define $A = \left[\frac{B}{C}\right]$. Then $A = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix}$, and $\mathcal{N}(A) = \mathcal{N}(B) \cap \mathcal{N}(C)$.

We obtain the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\begin{cases} x_1 & & - & x_3 & & + & 3x_5 & = & 0 \\ & x_2 & + & 4x_3 & & - & x_5 & = & 0 \\ & & & & x_4 & - & 2x_5 & = & 0 \\ & & & & & & 0 & = & 0 \end{cases}$$

A basis for $\mathcal{N}(A)$ (which is $\mathcal{N}(B) \cap \mathcal{N}(C)$) is constituted by the vectors $\mathbf{u}_1, \mathbf{u}_2$, in which $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$.

7. **Question.**

Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_k, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_\ell$ are vectors in $\mathbb{R}^n$, and $V = \mathsf{Span}\,(\{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_k\})$, $W = \mathsf{Span}\,(\{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_\ell\})$.

How do we find a basis for the subspace $V \cap W$ of $\mathbb{R}^n$?

Answer.

First recall the result ($\star$) below, proved in the handout *How to express the column space of a matrix as the null space of some matrix*:

($\star$) *Let* $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_q \in \mathbb{R}^n$, *and* $Y = [\ \mathbf{y}_1 \mid \mathbf{y}_2 \mid \cdots \mid \mathbf{y}_q\ ]$.

Denote by Y' *the reduced row-echelon form which is row-equivalent to* Z. *Denote the rank of* Y' *by* r, *and suppose* $0 < r < q$. *Write* $m = n - r$.

Suppose D *is a non-singular and invertible* $(n \times n)$*-matrix which satisfies* $Y' = DY$.

Denote by $D_\natural$ *the* $(m \times n)$*-matrix constituted by the bottom* m *rows of* D.

Then $\mathsf{Span}\,(\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_q\}) = \mathcal{C}(Y) = \mathcal{N}(D_\natural)$.

According to the result $(\star)$, there exist some matrices $B_\natural, C_\natural$, each with n columns, such that $V = \mathcal{N}(B_\natural)$ and $V \cap W = \mathcal{N}(C_\natural)$.

According to Lemma (2), $V \cap W = \mathcal{N}(A)$, in which $A = \left[\begin{array}{c} B_\sharp \\ \hline C_\natural \end{array}\right]$.

Then a basis for $V \cap W$ (which is regarded as the null space of A) can be obtained, as guaranteed by Theorem (D).

This answer actually provides an 'algorithm' that can be used in calculations.

8. **'Algorithm' for determining a basis for the intersection of two given spans of vectors.**

Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_k, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_\ell$ are vectors in $\mathbb{R}^n$, and $V = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_k\})$, $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_\ell\})$

We proceed to determine a basis for $V \cap W$ as described below:

- **Step (1).**
  Form $S = [\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \cdots \mid \mathbf{s}_k\ ]$. Further form the matrix $[\ S \mid I_n\ ]$.
  Apply row operations on $[\ S \mid I_n\ ]$ so as to result in the matrix $[\ S' \mid B\ ]$, which is row-equivalent to $[\ S \mid I_n\ ]$, and in which S' is the reduced row-echelon form row-equivalent to S.
- **Step (2).**
  Form $T = [\ \mathbf{t}_1 \mid \mathbf{t}_2 \mid \cdots \mid \mathbf{t}_\ell\ ]$. Further form the matrix $[\ T \mid I_n\ ]$.
  Apply row operations on $[\ T \mid I_n\ ]$ so as to result in the matrix $[\ T' \mid C\ ]$, which is row-equivalent to $[\ T \mid I_n\ ]$, and in which T' is the reduced row-echelon form row-equivalent to T.
- **Step (3).**
  Inspect the matrices S'. Denote the rank of S' by r_1.
  - $*$ Suppose $r_1 = n$. Then $V = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_k\}) = \mathbb{R}^n$, and $V \cap W = W$.
    Label the pivot columns of T', from left to right, by $d_1, d_2, \cdots, d_{r_2}$.
    A basis for $V \cap W$ (regarded as W) is constituted by $\mathbf{t}_{d_1}, \mathbf{t}_{d_2}, \cdots, \mathbf{t}_{d_{r_2}}$.
  - $*$ If $r_1 < n$, then proceed to Step (4).
- **Step (4).**
  From now on we are supposing $r_1 < n$.
  Inspect the matrices T'. Denote the rank of T' by r_2.
  - $*$ Suppose $r_2 = n$. Then $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_\ell\}) = \mathbb{R}^n$, and $V \cap W = V$.
    Label the pivot columns of S', from left to right, by $d_1^*, d_2^*, \cdots, d_{r_1}^*$.
    A basis for $V \cap W$ (regarded as V) is constituted by $\mathbf{s}^*_{d_1}, \mathbf{s}_{d_2^*}, \cdots, \mathbf{s}_{d_{r_1}^*}$.
  - $*$ If $r_2 < n$, then proceed to Step (5).
- **Step (5).**
  From now on we are supposing $r_1 < n$ and $r_2 < n$.
  Write $m_1 = n - r_1$ and $m_2 = n - r_2$.
  Denote by $B_\natural$ the $(m_1 \times n)$-matrix given by the bottom m_1 rows of B. (We have $V = \mathcal{N}(B_\natural)$.)
  Denote by $C_\natural$ the $(m_2 \times n)$-matrix given by the bottom m_2 rows of C. (We have $W = \mathcal{N}(C_\natural)$.)
  Form the $((m_1 + m_2) \times n)$-matrix A by $A = \left[\begin{array}{c} B_\natural \\ \hline C_\natural \end{array}\right]$. (We have $V \cap W = \mathcal{N}(B_\natural) \cap \mathcal{N}(C_\natural) = \mathcal{N}(A)$.)
  Obtain a basis for $V \cap W$, which is regarded as $\mathcal{N}(A)$, through, say, obtaining the reduced row-echelon form A' which is row-equivalent to A, and apply Theorem (D).
  (It can happen that $\mathcal{N}(A) = \{\mathbf{0}_n\}$. In this situation, the one and only one basis for $V \cap W$ is the empty set.)

9. **Illustrations on how to determine a basis for the intersection of two given spans of vectors.**

(a) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Define $V = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2\})$, $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2\})$.

We want to find a basis for $V \cap W$.

- Define $S = [\ \mathbf{s}_1 \mid \mathbf{s}_2\ ]$, $T = [\ \mathbf{t}_1 \mid \mathbf{t}_2\ ]$.

- We apply successive row operations starting from $[\ S \mid I_3\ ]$, in such a way to obtain some matrix $[\ S' \mid B\ ]$ in which S' is the reduced row-echelon form which is row equivalent to S:

$$[\ S \mid I_3\ ] = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right] = [\ S' \mid B\ ]$$

in which $S' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

The rank of S' is 2.

Define $B_\natural = [\ -1 \quad 1 \quad 1\ ]$. We have $\mathcal{N}(B_\natural) = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2\}) = V$.

- We apply successive row operations starting from $[\ T \mid I_3\ ]$, in such a way to obtain some matrix $[\ T' \mid C\ ]$ in which T' is the reduced row-echelon form which is row equivalent to T:

$$[\ T \mid I_3\ ] = \left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cc|ccc} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] = [\ T' \mid C\ ]$$

in which $T' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

The rank of T' is 2.

Define $C_\natural = [\ 0 \quad -1 \quad 1\ ]$. We have $\mathcal{N}(C_\natural) = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2\}) = W$.

- Define $A = \left[\begin{array}{c} B_\natural \\ \hline C_\natural \end{array}\right]$. We have $A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$. We have $\mathcal{N}(A) = \mathcal{N}(B_\sharp) \cap \mathcal{N}(C_\sharp) = V \cap W$.

We find the reduced row-echelon form A' which is row-equivalent to A:

$$A \longrightarrow \cdots\cdots \to A' = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

$\mathcal{LS}(A', \mathbf{0})$ reads as:

$$\left\{\begin{array}{rcrcrcl} x_1 & & & - & 2x_3 & = & 0 \\ & & x_2 & - & x_3 & = & 0 \end{array}\right.$$

Then a basis for $\mathcal{N}(A)$ (which is $V \cap W$) is constituted by $\mathbf{u}$, in which $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

(b) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

Define $V = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$, $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\})$.

We want to find a basis for $V \cap W$.

- Define $S = [\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \mathbf{s}_3\ ]$, $T = [\ \mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{t}_3\ ]$.
- We apply successive row operations starting from $[\ S \mid I_4\ ]$, in such a way to obtain some matrix $[\ S' \mid B\ ]$ in which S' is the reduced row-echelon form which is row equivalent to S:

$$[\ S \mid I_4\ ] = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] = [\ S' \mid B\ ]$$

in which $S' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

The rank of S' is 3.

Define $B_\natural = [\ 1 \quad -1 \quad -1 \quad 1\ ]$. We have $\mathcal{N}(B_\natural) = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}) = V$.

- We apply successive row operations starting from $[\ T \mid I_4\ ]$, in such a way to obtain some matrix $[\ T' \mid C\ ]$ in which T' is the reduced row-echelon form which is row equivalent to T:

$$[\ T \mid I_4\ ] = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] = [\ T' \mid C\ ]$$

in which $T' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

The rank of T' is 3.

Define $C_\natural = [\ 1 \quad 1 \quad 1 \quad 1\ ]$. We have $\mathcal{N}(C_\natural) = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}) = W$.

- Define $A = \left[\begin{array}{c} B_\natural \\ \hline C_\natural \end{array}\right]$. We have $A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. We have $\mathcal{N}(A) = \mathcal{N}(B_\sharp) \cap \mathcal{N}(C_\sharp) = V \cap W$.

  We find the reduced row-echelon form A' which is row-equivalent to A:

$$A \longrightarrow \cdots\cdots \to A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

  $\mathcal{LS}(A', \mathbf{0})$ reads as:

$$\begin{cases} x_1 & & & + & x_4 & = & 0 \\ & x_2 & + & x_3 & & = & 0 \end{cases}$$

  Then a basis for $\mathcal{N}(A)$ (which is $V \cap W$) is constituted by $\mathbf{u}_1, \mathbf{u}_2$, in which $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

Define $V = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2\})$, $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2\})$.

We want to find a basis for $V \cap W$.

- Define $S = [\ \mathbf{s}_1 \mid \mathbf{s}_2\ ]$, $T = [\ \mathbf{t}_1 \mid \mathbf{t}_2\ ]$.
- We apply successive row operations starting from $[\ S \mid I_4\ ]$, in such a way to obtain some matrix $[\ S' \mid B\ ]$ in which S' is the reduced row-echelon form which is row equivalent to S:

$$[\ S \mid I_4\ ] = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] = [\ S' \mid B\ ]$$

  in which $S' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

  The rank of S' is 2.

  Define $B_\natural = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. We have $\mathcal{N}(B_\natural) = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2\}) = V$.

- We apply successive row operations starting from $[\ T \mid I_4\ ]$, in such a way to obtain some matrix $[\ T' \mid C\ ]$ in which T' is the reduced row-echelon form which is row equivalent to T:

$$[\ T \mid I_4\ ] = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] = [\ T' \mid C\ ]$$

  in which $T' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

  The rank of T' is 2.

  Define $C_\natural = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. We have $\mathcal{N}(C_\natural) = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2\}) = W$.

- Define $A = \left[\begin{array}{c} B_\natural \\ \hline C_\natural \end{array}\right]$. We have $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. We have $\mathcal{N}(A) = \mathcal{N}(B_\sharp) \cap \mathcal{N}(C_\sharp) = V \cap W$.

  We find the reduced row-echelon form A' which is row-equivalent to A:

$$A \longrightarrow \cdots\cdots \to A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  $\mathcal{LS}(A', \mathbf{0})$ reads as:

$$\begin{cases} x_1 & & & + & x_4 & = & 0 \\ & x_2 & & & & = & 0 \\ & & x_3 & & & = & 0 \end{cases}$$

Then a basis for $\mathcal{N}(A)$ (which is $V \cap W$) is constituted by $\mathbf{u}$, in which $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(d) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

Define $V = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2\})$, $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2\})$.

We want to find a basis for $V \cap W$.

- Define $S = [\ \mathbf{s}_1 \mid \mathbf{s}_2\ ]$, $T = [\ \mathbf{t}_1 \mid \mathbf{t}_2\ ]$.
- We apply successive row operations starting from $[\ S \mid I_4\ ]$, in such a way to obtain some matrix $[\ S' \mid B\ ]$ in which S' is the reduced row-echelon form which is row equivalent to S:

$$[\ S \mid I_4\ ] = \left[\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cc|cccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right] = [\ S' \mid B\ ]$$

in which $S' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

The rank of S' is 2.

Define $B_\natural = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$. We have $\mathcal{N}(B_\natural) = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2\}) = V$.

- We apply successive row operations starting from $[\ T \mid I_4\ ]$, in such a way to obtain some matrix $[\ T' \mid C\ ]$ in which T' is the reduced row-echelon form which is row equivalent to T:

$$[\ T \mid I_4\ ] = \left[\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cc|cccc} 1 & 0 & -1/2 & 0 & 0 & -1/2 \\ 0 & 1 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] = [\ T' \mid C\ ]$$

in which $T' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -1/2 & 0 & 0 & -1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

The rank of T' is 2.

Define $C_\natural = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. We have $\mathcal{N}(C_\natural) = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2\}) = W$.

- Define $A = \left[\begin{array}{c} B_\natural \\ \hline C_\natural \end{array}\right]$. We have $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. We have $\mathcal{N}(A) = \mathcal{N}(B_\sharp) \cap \mathcal{N}(C_\sharp) = V \cap W$.

We find the reduced row-echelon form A' which is row-equivalent to A:

$$A \longrightarrow \cdots\cdots \rightarrow A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$\mathcal{LS}(A', \mathbf{0})$ reads as:

$$\begin{cases} x_1 & & & & = & 0 \\ & x_2 & & & = & 0 \\ & & x_3 & & = & 0 \\ & & & x_4 & = & 0 \end{cases}$$

Then $V \cap W = \mathcal{N}(A) = \{\mathbf{0}\}$, and its basis is given by the empty set.