MATH1030 More on minimal spanning sets.

1. Recall Theorem (E) from the handout *Minimal spanning set*:

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *be vectors in* $\mathbb{R}^n$, *and* U *be the* $(n \times q)$*-matrix given by* $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.
   Let $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.
   Denote by U' *the reduced row-echelon form which is row-equivalent to* U. *Denote the* j*-th column of* U' *by* $\mathbf{u}'_j$.
   Denote the rank of U' *by* r. *Suppose* $r \geq 1$, *and label the pivot columns of* U', *from left to right, by* $d_1, d_2, \cdots, d_r$.
   Then $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ *constitute a basis for* V.
   Moreover, for each $j = 1, 2, \cdots, q$, *the vector* $\mathbf{u}_j$ *is the linear combination of* $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ *and the respective scalars* $\alpha_1, \alpha_2, \cdots, \alpha_r$ *if and only if* $\mathbf{u}'_j = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

   We give some applications of Theorem (E) in the theory for the notion of *basis*.

2. **Lemma (1).**
   Let V, W *be subspaces of* $\mathbb{R}^n$.
   Define the set $V + W$ *by* $V + W = \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{l} \text{There exist some } y \in V,\ z \in W \\ \text{such that } x = y + z \end{array} \right\}$.
   Then $V + W$ *is a subspace of* $\mathbb{R}^n$.

   Remark. $V + W$ is called the sum of V and W.

   Proof of Lemma (1). Exercise.

3. An immediate application of Theorem (E) in helping us determine a basis for the sum of two subspaces of $\mathbb{R}^n$ when a basis for each subspace concerned is already known (or, more generally, when each subspace concerned has already been expressed as the span of several vectors in $\mathbb{R}^n$).

   Theorem (2).
   Let V, W *be subspaces of* $\mathbb{R}^n$. *Let* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in V$, $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots u_{p+q} \in W$. *Suppose none of these vectors is the zero vector.*
   Suppose $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\})$ *and* $W = \mathsf{Span}\ (\{\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\})$.
   Then the statements below hold:

   (a) $V + W = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\})$.

   (b) *Suppose* $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_p \mid \mathbf{u}_{p+1} \mid \mathbf{u}_{p+2} \mid \cdots \mid \mathbf{u}_{p+q}\ ]$. *Denote by* U' *the reduced row-echelon form which is row-equivalent to* U. *Denote the rank of* U' *by* r. *Label the pivot columns of* U' *from left to right by* $d_1, d_2, \cdots, d_r$.
   Then a basis for $V + W$ *is constituted by* $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$.

   (c) *Suppose* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *constitute a basis for* V, *and* $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}$ *constitute a basis for* W.
   Then $d_j = j$ *for each* $j = 1, 2, \cdots, p$. *Moreover,* $r \geq p$.
   Further write $s = r - p$, *and* $k_1 = d_{p+1} - p$, $k_2 = d_{p+2} - p$, ..., $k_s = d_r - p$.
   Then a basis for $V + W$ *is constituted by* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{u}_{p+k_1}, \mathbf{u}_{p+k_2}, \cdots, \mathbf{u}_{p+k_s}$.

   Proof of Theorem (2). Exercise. (Apply Lemma (1) and Theorem (E). The hard work has been done in the proof of Theorem (E).)

4. **Illustrations for Theorem (2).**

   (a) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} -1 \\ 1 \\ 5 \\ 5 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix}$, $\mathbf{t}_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$.
   Define $V = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$, $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\})$.
   We want to find a basis for $V + W$.

Define $U = [\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \mathbf{s}_3 \mid \mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{t}_3 \mid \mathbf{t}_4\ ]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 1 & 2 & 7 & -1 & 1 & -1 & 3 \\ 1 & 1 & 3 & 1 & 1 & 0 & 2 \\ 3 & 2 & 5 & 5 & -1 & 9 & 1 \\ 1 & -1 & -5 & 5 & 2 & 0 & 1 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & -1 & 3 & 0 & 3 & 0 \\ 0 & 1 & 4 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of U' is 3, and the pivot columns are the first, second, and fifth columns.

Hence a basis for $V + W$ is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_2$.

(b) *Let* $\mathbf{s}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} -3 \\ -4 \\ 0 \\ -6 \\ 1 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 3 \\ 8 \\ 2 \\ 5 \\ -1 \end{bmatrix}$.

Define $V = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$, $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\})$.

We want to find a basis for $V + W$.

Define $U = [\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \mathbf{s}_3 \mid \mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{t}_3\ ]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} -1 & 1 & -3 & 1 & 1 & 3 \\ 1 & -2 & -4 & 0 & 1 & 8 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ -2 & 3 & -6 & 2 & 2 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The rank of U' is 5, and the pivot columns are the first five columns.

Hence $V + W = \mathbb{R}^5$, and a basis for $V + W$ is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{t}_1, \mathbf{t}_2$.

(c) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -3 \\ 3 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} 7 \\ -12 \\ 5 \\ -12 \\ 9 \end{bmatrix}$, $\mathbf{s}_4 = \begin{bmatrix} -3 \\ 4 \\ -1 \\ 4 \\ -5 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 2 \\ -4 \\ 1 \\ -5 \\ 6 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 4 \\ -7 \\ 3 \\ -7 \\ 5 \end{bmatrix}$, $\mathbf{t}_4 = \begin{bmatrix} 2 \\ -5 \\ 2 \\ -3 \\ 3 \end{bmatrix}$,

$\mathbf{t}_5 = \begin{bmatrix} 6 \\ -9 \\ 4 \\ -10 \\ 7 \end{bmatrix}$, $\mathbf{t}_6 = \begin{bmatrix} 4 \\ -7 \\ 3 \\ -6 \\ 5 \end{bmatrix}$.

Define $V = \mathsf{Span}\ (\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4\})$, $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6\})$.

We want to find a basis for $V + W$.

Define $U = [\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \mathbf{s}_3 \mid \mathbf{s}_4 \mid \mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{t}_3 \mid \mathbf{t}_4 \mid \mathbf{t}_5 \mid \mathbf{t}_6\ ]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 1 & 2 & 7 & -3 & 1 & 2 & 4 & 2 & 6 & 4 \\ -2 & -3 & -12 & 4 & 0 & -4 & -7 & -5 & -9 & -7 \\ 1 & 1 & 5 & -1 & 0 & 1 & 3 & 2 & 4 & 3 \\ -2 & -3 & -12 & 4 & -1 & -5 & -7 & -3 & -10 & -6 \\ 1 & 3 & 9 & -5 & 1 & 6 & 5 & 3 & 7 & 5 \end{bmatrix}$$

$$\longrightarrow \cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & -1 & 2 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of U' is 4, and the pivot columns are the first, second, fifth, eighth columns.

Hence a basis for $V + W$ is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_1, \mathbf{t}_4$.

Remark. Suppose V, W are respectively given as the null spaces of some matrices with n columns. Then we first obtain a basis for V and a basis for W, and then apply Theorem (2) to obtain a basis for $V + W$.

5. **Theorem (F). (Replacement Theorem in the context of subspaces of $\mathbb{R}^n$.)**

Let W be a subspace of $\mathbb{R}^n$.

Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in W. Suppose none of these vectors is the zero vector.

Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ are linearly independent.

Further suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for W.

Then, $q \geq p$, and there is a basis for W which is constituted by $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ together with some $q - p$ vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$.

Remark. In plain words, the conclusion in this result says that *the linearly independent vectors $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ in W (which do not necessarily constitute a basis for W because there may be not enough of them to 'span' every vector in W) can be used for 'replacing' p vectors from amongst any given basis for W, say, $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$.*

6. **Proof of Theorem (F).**

   Let W be a subspace of $\mathbb{R}^n$.

   Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in W. Suppose none of these vectors is the zero vector.

   Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ are linearly independent.

   Further suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for W.

   Write $\mathbf{u}_k = \mathbf{s}_k$ for each $k = 1, 2, \cdots, p$, and write $\mathbf{u}_{p+\ell} = \mathbf{t}_\ell$ for each $\ell = 1, 2, \cdots, q$.

   Define $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\})$.

   By assumption, $W = \mathsf{Span}\ (\{\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\})$.

   We have $V + W = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\})$.

   Then, by assumption, $V + W = \mathsf{Span}\ (\{\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\}) = W$.

   The conclusion then follows from an application of Theorem (2).

7. **Illustrations for Theorem (F).**

   (a) *Let* $\mathbf{s}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

   Take for granted that $\mathbf{s}_1$, $\mathbf{s}_2$ are linearly independent, and that $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ constitute a basis for $\mathbb{R}^3$.

   We want to obtain a basis for $\mathbb{R}^3$ constituted by $\mathbf{s}_1, \mathbf{s}_2$ and some vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$.

   Define $U = [\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{t}_3\ ]$.

   We find the reduced row-echelon form U' which is row equivalent to U:

   $$U = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1/2 \end{bmatrix}$$

   The rank of U' is 3. The pivot columns of U' are the first, second and fourth columns.

   Hence a basis for $\mathbb{R}^3$ is constutitued by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_2$.

   (b) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$.

   Define $W = \mathsf{Span}\ (\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\})$.

   Take for granted that $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ are constitutes a basis for W.

   Note that $\mathbf{s}_1$ is linearly independent. Take for granted that $\mathbf{s}_1 \in W$.

   We want to obtain a basis for W constituted by $\mathbf{s}_1$ and some vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$.

   Define $U = [\ \mathbf{s}_1 \mid \mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{t}_3\ ]$.

   We find the reduced row-echelon form U' which is row equivalent to U:

   $$U = \begin{bmatrix} 1 & 2 & 7 & 1 \\ 1 & 1 & 3 & 1 \\ 3 & 2 & 5 & -1 \\ 1 & -1 & -5 & 2 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   The rank of U' is 3. The pivot columns of U' are the first, second and fourth columns.

   Hence a basis for W is constituted by $\mathbf{s}_1, \mathbf{t}_1, \mathbf{t}_3$.

   (c) *Let* $\mathbf{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, *and* $\mathbf{t}_j = \mathbf{e}_j^{(5)}$ *for each* $j = 1, 2, 3, 4, 5$.

   Take for granted that $\mathbf{s}_1$, $\mathbf{s}_2, \mathbf{s}_3$ are linearly independent, and that $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5$ constitute a basis for $\mathbb{R}^5$.

   We want to obtain a basis for $\mathbb{R}^5$ constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ and some vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5$.

   Define $U = [\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \mathbf{s}_3 \mid \mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{t}_3 \mid \mathbf{t}_4 \mid \mathbf{t}_5\ ]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} -2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & -4 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & 4 \end{bmatrix}$$

The rank of U' is 5. The pivot columns of U' are the first, second, third, fourth and fifth columns.
Hence a basis for $\mathbb{R}^5$ is constutitued by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{t}_1, \mathbf{t}_2$.

(d) *Let* $\mathbf{s}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{s}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, *and* $\mathbf{t}_j = \mathbf{e}_j^{(7)}$ *for each* $j = 1, 2, 3, 4, 5, 6, 7$.

Take for granted that $\mathbf{s}_1$, $\mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ *are linearly independent, and that* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6, \mathbf{t}_7$ *constitute a basis for* $\mathbb{R}^7$.
We want to obtain a basis for $\mathbb{R}^7$ *constituted by* $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ *and some vectors from amongst* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6, \mathbf{t}_7$.
Define $U = [\ \mathbf{s}_1 \mid \mathbf{s}_2 \mid \mathbf{s}_3 \mid \mathbf{s}_4 \mid \mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{t}_3 \mid \mathbf{t}_4 \mid \mathbf{t}_5 \mid \mathbf{t}_6 \mid \mathbf{y}_7\ ]$.
We find the reduced row-echelon form U' which is row equivalent to U:

$$\begin{aligned} U &= \begin{bmatrix} -4 & -2 & -1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 6 & -6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &\longrightarrow \cdots\cdots \longrightarrow U' \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -6 & 6 \end{bmatrix} \end{aligned}$$

The rank of U' is 7. The pivot columns of U' are the first, second, third, fourth, fifth, seventh and eighth columns.
Hence a basis for W is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_4$.

8. Recall the statement of Theorem (B) from the handout *Bases for subspaces of* $\mathbb{R}^n$:

   Any two bases for a subspace of $\mathbb{R}^n$ *have the same number of vectors.*

   Equipped with Theorem (F), we are now ready to prove Theorem (B).

9. **Proof of Theorem (B).**

   Let W be a subspace of $\mathbb{R}^n$.

   If W be the zero subspace, then the empty set is the only basis for W, and in this situation, there is nothing to prove.

   From now on suppose W is a non-zero subspace of $\mathbb{R}^n$.

   Suppose $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$ constitute a basis for W.

   Also suppose $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{p'}$ constitute a basis for W.

   We are going to verify that $p = p'$:

   - By assumption, $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$ are linearly independent vectors in W, and $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{p'}$ constitute a basis for W. Then $p \leq p'$.
     Also by assumption, $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{p'}$ are linearly independent vectors in W, and $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$constitute a basis for W. Then $p' \leq p$.
     It follows that $p = p'$.

10. Now recall Theorem (C) below, proved in the handout *Bases for subspaces of* $\mathbb{R}^n$:

    Suppose W *is a non-zero subspace of* $\mathbb{R}^n$. *Then* W *has a basis which consists of at least one and at most* n *vectors in* $\mathbb{R}^n$.

Combining Theorem (C) and Theorem (F), we will obtain Theorem (G).

11. **Theorem (G). (Extension of linearly independent set to basis in the context of $\mathbb{R}^n$.)**

Let W be a non-zero subspace of $\mathbb{R}^n$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in W.

Further suppose that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.

Then, there is some basis for W, which is constituted of at most n vectors, amongst them being the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.

Remark. In plain words, the conclusion in this result says that

> *the linearly independent vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ in W (which do not necessarily constitute a basis for W because there may be not enough of them to 'span' every vector in W) can be 'extended' to give a basis for W.*

12. **Proof of Theorem (G).**

By Theorem (C), W has a basis, constituted by, say, some q vectors $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}$ in W, for which $q \leq n$.

None of these q vectors is the zero vector.

By assumption, none of the p vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ is the zero vectors, and each of these vectors is a linear combination of $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}$.

Then by Theorem (F), we have $q \geq p$, and there is a basis for W which is constituted by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ together with some $q - p$ vectors from amongst $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}$.