MATH1030 Minimal spanning sets.

1. **Question** $(*)$.

   Given a number of vectors in $\mathbb{R}^n$, can we extract from these vectors a basis for the span of these vectors?

   Answer to Question $(*)$.

   The answer is *yes*, and it is explained in full by Theorem (E).

2. **Theorem (E).**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be vectors in $\mathbb{R}^n$, and U be the $(n \times q)$-matrix given by $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

   Let $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

   Denote by U' the reduced row-echelon form which is row-equivalent to U. Denote the j-th column of U' by $\mathbf{u}'_j$.

   Denote the rank of U' by r. Suppose $r \geq 1$, and label the pivot columns of U', from left to right, by $d_1, d_2, \cdots, d_r$. Then $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ constitute a basis for V.

   Moreover, for each $j = 1, 2, \cdots, q$, the vector $\mathbf{u}_j$ is the linear combination of $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_r$ if and only if $\mathbf{u}'_j = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

   Remark. Once we make sense of the notion of *dimension*, it will turn that the dimension of V is r, because one base for V, namely, $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ is constituted by r vectors.

   Further remark. $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ can be thought of as constituting a 'minimal spanning set' for V in the sense that they span V, and any $r - 1$ or fewer vectors from amongst them definitely fail to span V.

3. In many situations we only need the validity of the result below, which is a consequence of Theorem (E), to affirm the existence of basis for a span of vectors.

   Corollary to Theorem (E).

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are non-zero vectors in $\mathbb{R}^n$. Then some vectors amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ constitute a basis for the span of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

4. **'Algorithm' associated with Theorem (E).**

   When a collection of vectors, say, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ in $\mathbb{R}^n$, are given to us in 'concrete' terms, Theorem (E) combined with what we know about solving equations, suggest an 'algorithm' for obtaining a basis for $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

   - **Step (1).**
     Form the matrix $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n\ ]$.
   - **Step (2).**
     Obtain the reduced row-echelon form U' which is row equivalent to U.
   - **Step (3).**
     Read off from U', the rank r of U', and the pivot columns of U'. Label the pivot columns of U', from left to right, by $d_1, d_2, \cdots, d_r$.
   - **Step (4).**
     Conclude that the vectors $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ a basis for V.

5. **Illustrations.**

   (a) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}$, *and $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$.*

   We want to find extract a basis from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ for V.

   Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.

   We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank of U' is 2, and the pivot columns are the first and second columns.
Hence a basis for V is constituted by $\mathbf{u}_1, \mathbf{u}_2$.
As a bonus, we also see that $\mathbf{u}_3 = \mathbf{u}_1 - 2\mathbf{u}_2$.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \mathbf{u}_6 = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$, *and* $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\})$.
We want to find extract a basis from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ *for* V.
Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6\ ]$.
We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}.$$

The rank of U' is 3, and the pivot columns are the first, second, and fourth columns.
Hence a basis for V is constituted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$.
As a bonus, we also see that $\mathbf{u}_3 = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_5 = \mathbf{u}_1 + \mathbf{u}_4$, and $\mathbf{u}_6 = 10\mathbf{u}_1 - 8\mathbf{u}_2 + 5\mathbf{u}_4$.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, *and* $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.
We want to find extract a basis from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *for* V.
Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.
We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 4 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of U' is 3, and the pivot columns are the first, second, and third columns.
Hence a basis for V is constituted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.
As a bonus, we also see that $\mathbf{u}_3 = \mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$.

(d) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix}$, *and* $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\})$.
We want to find extract a basis from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ *for* V.
Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ]$.
We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of U' is 3, and the pivot columns are the first, second, and forth columns.
Hence a basis for V is constituted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$.
As a bonus, we also see that $\mathbf{u}_3 = -\mathbf{u}_1 + 4\mathbf{u}_2$, $\mathbf{u}_5 = 3\mathbf{u}_1 - \mathbf{u}_2 - 2\mathbf{u}_4$.

(e) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ 6 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 3 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \mathbf{u}_6 = \begin{bmatrix} -7 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{u}_7 = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}$, *and* $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{u}_7\})$.
We want to find extract a basis from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{u}_7$ *for* V.
Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6 \mid \mathbf{u}_7\ ]$.
We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of U' is 3, and the pivot columns are the first, third, and fourth columns.
Hence a basis for V is constituted by $\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_4$.
As a bonus, we also see that $\mathbf{u}_2 = -2\mathbf{u}_1$, $\mathbf{u}_5 = \mathbf{u}_3 + \mathbf{u}_4$, $\mathbf{u}_6 = \mathbf{u}_1 - 2\mathbf{u}_3 - \mathbf{u}_4$, $\mathbf{u}_7 = \mathbf{u}_1 + 3\mathbf{u}_3 + 2\mathbf{u}_4$.

6. **Preparation towards proving Theorem (E).**

Recall the result (I) below, from the handout *More on span of vectors and column space of a matrix*:

(I) *Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_k$ be vectors in $\mathbb{R}^n$. The statements below are logically equivalent:*

   (a) *Each of $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_k$ is a linear combination of $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$.*

   (b) $\mathsf{Span}\,(\{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_k\}) = \mathsf{Span}\,(\{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p\})$.

With the help of the result (I), we are going to prove Lemma (1), which looks trivial but is in fact the cornerstone for the proof of Theorem (E).

Lemma (1).

Let H be an $(n \times q)$-matrix. Suppose H is a reduced row-echelon form. Denote the rank of H by r.

(a) *Suppose $r = 0$. Then $\mathcal{C}(H) = \{\mathbf{0}_n\}$.*

(b) *Suppose $r \geq 1$. Then $\mathcal{C}(H) = \mathsf{Span}\,(\{\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_r^{(n)}\})$.*

Proof of Lemma (1).

Let H be an $(n \times q)$-matrix. Suppose H is a reduced row-echelon form. Denote the rank of H by r.

(a) Suppose $r = 0$. Then H is the zero matrix. Therefore $\mathcal{C}(H) = \{\mathbf{0}_n\}$.

(b) Suppose $r \geq 1$. Denote the columns of H, from left to right, by $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q$.

   Label the pivot columns of H, from left to right, by $d_1, d_2, \cdots, d_r$.

   The vectors $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_r^{(n)}$ are amongst $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q$; in fact they are respectively the d_1-th, d_2-th, ..., d_r-th columns of H.

   The bottom $n - r$ entries of each of $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q$ are all zero. Then each of them is a linear combination of $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_r^{(n)}$.

   It follows, from the result (I), that $\mathcal{C}(H) = \mathsf{Span}\,(\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\}) = \mathsf{Span}\,(\{\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_r^{(n)}\})$.

7. **Further preparation towards proving Theorem (E).**

Next recall the result (II) below, from the handout *Row operations and row equivalence in terms of multiplication by non-singular and invertible matrices*:

(II) *Let C, D be $(n \times q)$-matrices.*

   The statements below are logically equivalent:

   (a) *C is row-equivalent to D.*

   (b) *There exists some non-singular and invertible $(n \times n)$-square matrix A such that $D = AC$.*

Also recall the collection of results (III.1), (III.2), (III.3) from the handouts *Linear combinations*, *More on span of vectors and column space of a matrix*, *More on linear dependence and linear independence*:

(III.1) *Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}$ be vectors in $\mathbb{R}^n$ and $\alpha_1, \alpha_2, \cdots, \alpha_p$ be real numbers.*

   Suppose A is a non-singular $(n \times n)$-square matrix. Then the statements below are logically equivalent:

   (a) *$\mathbf{t}$ is a linear combination of the vectors $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_p$.*

   (b) *$A\mathbf{t}$ is a linear combination of the vectors $A\mathbf{s}_1, A\mathbf{s}_2, \cdots, A\mathbf{s}_p$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_p$.*

(III.2) *Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in $\mathbb{R}^n$.*

   Suppose A is a non-singular $(n \times n)$-square matrix. Then the statements below are logically equivalent:

   (a) $\mathsf{Span}\,(\{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p\}) = \mathsf{Span}\,(\{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q\})$.

   (b) $\mathsf{Span}\,(\{A\mathbf{s}_1, A\mathbf{s}_2, \cdots, A\mathbf{s}_p\}) = \mathsf{Span}\,(\{A\mathbf{t}_1, A\mathbf{t}_2, \cdots, A\mathbf{t}_q\})$.

(III.3) *Let $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in $\mathbb{R}^n$.*

   Suppose A is a non-singular $(n \times n)$-square matrix. Then the statements below are logically equivalent:

   (a) *$\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ are linearly independent.*

   (b) *$A\mathbf{t}_1, A\mathbf{t}_2, \cdots, A\mathbf{t}_q$ are linearly independent.*

Equipped with Lemma (1) and the results (II), (III.1) (II.2), (III.3), we are ready to prove Theorem (E).

8. **Proof of Theorem (E).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be vectors in $\mathbb{R}^n$, and U be the $(n \times q)$-matrix given by $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

Let $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

Denote by U' the reduced row-echelon form which is row-equivalent to U. Denote the j-th column of U' by $\mathbf{u}'_j$.

Denote the rank of U' by r. Suppose $r \geq 1$, and label the pivot columns of U', from left to right, by $d_1, d_2, \cdots, d_r$.

According to the result (II), there exists some non-singular and invertible $(n \times n)$-square matrix A such that $U' = AU$.

For the same matrix A, we have $\mathbf{u}'_j = A\mathbf{u}_j$ for each $j = 1, 2, \cdots, q$.

- [We verify that $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_k}$ are linearly independent.]
  Note that $\mathbf{u}'_{d_k} = \mathbf{e}_k^{(n)}$ for each $k = 1, 2, \cdots, r$. Then $\mathbf{u}'_{d_1}, \mathbf{u}'_{d_2}, \cdots, \mathbf{u}'_{d_k}$ are linearly independent.
  Since A is a non-singular and invertible matrix, $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_k}$ are linearly independent, according to the result (III.3).
- [We verify that $V = \mathsf{Span}\ (\{\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_k}\})$.]
  Since $r \geq 1$, U' is not the zero matrix.
  By Lemma (1), we have

  $$\mathsf{Span}\ \{\mathbf{u}'_1, \mathbf{u}'_2, \cdots, \mathbf{u}'_q\} = \mathcal{C}(U') = \mathsf{Span}\ (\{\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_r^{(n)}\}) = \mathsf{Span}\ (\{\mathbf{u}'_{d_1}, \mathbf{u}'_{d_2}, \cdots, \mathbf{u}'_{d_r}\}).$$

  Since A is a non-singular $(n \times n)$-square matrix, $V = \mathsf{Span}\ \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\} = \mathsf{Span}\ (\{\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}\})$, according to the result (III.2).

It follows that $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ constitute a basis for V.

Let $j = 1, 2, \cdots, q$. According to the result (III.1), the statements below are logically equivalent:

(a) $\mathbf{u}_j$ is the linear combination of $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_r$.

(b) $\mathbf{u}'_j$ is the linear combination of $\mathbf{u}'_{d_1}, \mathbf{u}'_{d_2}, \cdots, \mathbf{u}'_{d_r}$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_r$.

Recall that $\mathbf{u}'_{d_k} = \mathbf{e}_k^{(m)}$. Hence the latter statement is equivalent to the equality $\mathbf{u}'_j = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

9. The results (II.1), (II.3) also combine immediately to give Theorem (2), whose proof is left as an exercise. (Imitate the argument for Theorem (E).) Another argument for Theorem (2) is given by an application of Theorem (E). (Start by asking what the reduced row echelon form which is row-equivalent to the matrix $[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$, and what that is to the matrix $[\ A\mathbf{u}_1 \mid A\mathbf{u}_2 \mid \cdots \mid A\mathbf{u}_q\ ]$)

Theorem (2).

Let V be a subspace of $\mathbb{R}^n$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be vectors in V.

Let A be a non-singular and invertible $(n \times n)$-square matrix, and $W = \mathsf{Span}\ \{A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_q\}$.

Let p be an integer between 1 and q. Suppose $j_1, j_2, \cdots, j_p$ are integers between 1 and q, and satisfy $1 \leq j_1 < j_2 < \cdots < j_p \leq q$. Then the statements below are logically equivalent:

(a) $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \cdots, \mathbf{u}_{j_p}$ *constitute a basis for V.*

(b) $A\mathbf{u}_{j_1}, A\mathbf{u}_{j_2}, \cdots, A\mathbf{u}_{j_p}$ *constitute a basis for W.*