MATH1030 How to express the column space of a matrix as the null space of some matrix.

1. Recall an observation from the handout *Homogeneous systems and null spaces*:

   Suppose we are given an $(m \times n)$ matrix $B$.
   To determine $\mathcal{N}(B)$ is the same as giving an 'explicit' description of the solution set of the homogeneous system $\mathcal{LS}(B, \mathbf{0})$ through set language, in terms of (hopefully just a few) solutions of the system. That amounts to finding all solutions of $\mathcal{LS}(B, \mathbf{0})$.

   In practice, this is what we proceed with the above:

   Suppose $B'$ is the reduced row-echelon form which is row-equivalent to $B$.
   Suppose the rank of $B'$ is $r$. Write $k = n - r$.
   When $k = 0$, $\mathcal{N}(B) = \{0\}$.
   Suppose $k > 0$. Then those (few) solutions $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ of $\mathcal{LS}(B, \mathbf{0})$ needed for expressing all solutions of $\mathcal{LS}(B, \mathbf{0})$ are 'read off' as solutions of $\mathcal{LS}(B', \mathbf{0})$ for which one free variable takes the value 1 and all other free variable take the value 0.
   In conclusion we have

   $$\mathcal{N}(B) = \mathcal{N}(B') = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid c_1, c_2, \cdots, c_k \in \mathbb{R}\} = \mathsf{Span}\ (\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}).$$

   A natural follow-up question is: *can this process be reversed?* (And in what sense can this be reversed?)

2. **Question.**

   Suppose we are given a collection of vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ in $\mathbb{R}^n$.

   Can we express $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$ as the null space of some appropriate matrix with $n$ columns?

   **Answer.**

   The answer is '*yes*', and will be provided by Theorem (M).

   **Remark.** Hence, the null space of a matrix is the span of some vectors, while the span of several vectors is the null space of some matrix. The notions of *null space*, *span*, *column space* are manifestations of the same mathematical concept.

3. **Theorem (M).**

   *Let* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^n$, *and* $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

   *Denote by* $U'$ *the reduced row-echelon form which is row-equivalent to* $U$. *Denote the rank of* $U'$ *by* $r$, *and suppose* $0 < r < q$. *Write* $p = n - r$.

   *Suppose* $A$ *is a non-singular and invertible* $(n \times n)$*-matrix which satisfies* $U' = AU$.

   *Denote by* $A_\natural$ *the* $(p \times n)$*-matrix constituted by the bottom* $p$ *rows of* $A$.

   *Then* $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \mathcal{C}(U) = \mathcal{N}(A_\natural)$.

   **Remarks on the statement of Theorem (M).**

   (a) Theorem (M) is meaningful (and useful) because of the validity of the result $(\star)$ below from the handout *Row equivalence in terms of multiplication by non-singular and invertible matrices*:

   $(\star)$ *Let* $C, D$ *be* $(n \times q)$*-matrices.*
   *The statements below are logically equivalent:*
   i. $C$ *is row-equivalent to* $D$.
   ii. *There exists some non-singular and invertible* $(n \times n)$*-square matrix* $A$ *such that* $D = AC$.

   (b) Theorem (M) is formulated in such a way to avoid the complications in having to cover the 'extreme cases' '$r = 0$', '$r = n$' within the statement.

   i. When $r = 0$, we have $U = \mathcal{O}_{n\times q}$ and $\mathcal{C}(U) = \{\mathbf{0}_n\} = \mathcal{N}(I_n)$.
   ii. When $r = n$, we have $\mathcal{C}(U) = \mathcal{C}(I_n) = \mathbb{R}^n = \mathcal{N}(\mathcal{O}_{1\times n})$.

4. **Proof of Theorem (M).**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^n$, and $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$. We have $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \mathcal{C}(U)$.

   Denote by $U'$ the reduced row-echelon form which is row-equivalent to $U$. Denote the rank of $U'$ by $r$, and suppose $0 < r < q$. Write $p = n - r$.

   Suppose $A$ is a non-singular and invertible $(n \times n)$-matrix which satisfies $U' = AU$.

   Denote by $A_\natural$ the $(p \times n)$-matrix constituted by the bottom $p$ rows of $A$.

   Denote by $A_\sharp$ the $(r \times n)$-matrix constituted by the top $r$ rows of $A$. So $A = \left[\begin{array}{c} A_\sharp \\ \hline A_\natural \end{array}\right]$.

Denote by $U'_\sharp$ the $(r \times q)$-matrix constituted by the top $r$ rows of $U'$. So $U' = \left[\begin{array}{c} U'_\sharp \\ \hline \mathcal{O}_{p\times q}\end{array}\right]$.

We want to verify that $\mathcal{C}(U) = \mathcal{N}(A_\natural)$.

- [We verify that every vector in $\mathcal{C}(U)$ belongs to $\mathcal{N}(A_\natural)$.
  This amounts to verify the statement '*For any* $\mathbf{t} \in \mathbb{R}^n$, *if* $\mathbf{t} \in \mathcal{C}(U)$ *then* $\mathbf{t} \in \mathcal{N}(A_\natural)$'.]
  Pick any $\mathbf{t} \in \mathbb{R}^n$.
  Suppose $\mathbf{t} \in \mathcal{C}(U)$. Then there exists some $\mathbf{z} \in \mathbb{R}^q$ such that $\mathbf{t} = U\mathbf{z}$.
  We have $\left[\begin{array}{c} U'_\sharp\mathbf{z} \\ \hline \mathbf{0}_p\end{array}\right] = \left[\begin{array}{c} U'_\sharp\mathbf{z} \\ \hline \mathcal{O}_{p\times q}\mathbf{z}\end{array}\right] = \left[\begin{array}{c} U'_\sharp \\ \hline \mathcal{O}_{p\times q}\end{array}\right]\mathbf{z} = U'\mathbf{z} = AU\mathbf{z} = A\mathbf{t} = \left[\begin{array}{c} A_\sharp \\ \hline A_\natural\end{array}\right]\mathbf{t} = \left[\begin{array}{c} A_\sharp\mathbf{t} \\ \hline A_\natural\mathbf{t}\end{array}\right]$.
  Then $A_\natural\mathbf{t} = \mathbf{0}_p$.
  Therefore $\mathbf{t} \in \mathcal{N}(A_\natural)$.
- [We verify that every vector in $\mathcal{N}(A_\natural)$ belongs to $\mathcal{C}(U)$.
  This amounts to verify the statement '*For any* $\mathbf{t} \in \mathbb{R}^n$, *if* $\mathbf{t} \in \mathcal{N}(A_\natural)$ *then* $\mathbf{t} \in \mathcal{C}(U)$'.]
  Pick any $\mathbf{t} \in \mathbb{R}^n$.
  Suppose $\mathbf{t} \in \mathcal{N}(A_\natural)$. Then $A_\natural\mathbf{t} = \mathbf{0}_p$.
  We have $A\mathbf{t} = \left[\begin{array}{c} A_\sharp \\ \hline A_\natural\end{array}\right]\mathbf{t} = \left[\begin{array}{c} A_\sharp\mathbf{t} \\ \hline A_\natural\mathbf{t}\end{array}\right] = \left[\begin{array}{c} A_\sharp\mathbf{t} \\ \hline \mathbf{0}_p\end{array}\right]$.
  Consider the system $\mathcal{LS}(U, \mathbf{t})$. Its augmented matrix representation is $[\ U \mid \mathbf{t}\ ]$,
  Since $A$ is non-singular, $[\ U \mid \mathbf{t}\ ]$ is row-equivalent to the matrix $A[\ U \mid \mathbf{t}\ ]$, which is explicitly given by

  $$A[\ U \mid \mathbf{t}\ ] = [\ U' \mid A\mathbf{t}\ ] = \left[\begin{array}{c|c} U'_\sharp & A_\sharp\mathbf{t} \\ \hline \mathcal{O}_{p\times q} & A_\natural\mathbf{t}\end{array}\right] = \left[\begin{array}{c|c} U'_\sharp & A_\sharp\mathbf{t} \\ \hline \mathcal{O}_{p\times q} & \mathbf{0}_p\end{array}\right],$$

  which is a reduced row-echelon form whose last column is not a pivot column.
  Then the system $\mathcal{LS}(U, \mathbf{t})$ is consistent. Therefore there exists some $\mathbf{z} \in \mathbb{R}^q$ such that $U\mathbf{z} = \mathbf{t}$.
  Hence $\mathbf{t} \in \mathcal{C}(U)$.

It follows that $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \mathcal{C}(U) = \mathcal{N}(A_\natural)$.

5. Theorem (M) suggests an 'algorithm' with which we can express the span of some 'concretely' given vectors in $\mathbb{R}^n$ explicitly as the null space of a 'concretely' determined matrix with $n$ columns.

   **'Algorithm' associated with Theorem (M).**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^n$. We are going to write down a matrix with $n$ columns whose null space is the same as the span of these vectors.

   - **Step (0).**
     If $\mathbf{u}_1 = \mathbf{u}_2 = \cdots = \mathbf{u}_q = \mathbf{0}_n$ then $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \mathcal{N}(I_n)$.
     From now on assume $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are not all zero vectors.
   - **Step (1).**
     Form the matrix $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n\ ]$.
     Further form the matrix $[\ U \mid I_n\ ]$.
   - **Step (2).**
     Apply row operations on $[\ U \mid I_n\ ]$ so as to result in the matrix $[\ U' \mid A\ ]$, which is row-equivalent to $[\ U \mid I_n\ ]$, and in which $U'$ is the reduced row-echelon form row-equivalent to $U$.
   - **Step (3).**
     Inspect the matrix $U'$. Denote its rank by $r$.
     - Suppose $r = n$. Then $\mathsf{Span}\ (\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q) = \mathcal{N}(\mathcal{O}_{1\times n})$.
     - Suppose $r < n$. Write $p = n - r$. Denote by $A_\natural$ the $(p \times n)$-matrix given by the bottom $p$ rows of $A$. Then $\mathsf{Span}\ (\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q) = \mathcal{N}(A_\natural)$.

6. **Illustrations.**

   (a) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ -2 \\ 3\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -5\end{bmatrix}$.
   *We want to express* $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$ *as the null space of some appropriate matrix with three columns.*
   Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.
   We apply successive row operations starting from $[\ U \mid I_3\ ]$, in such a way to obtain some matrix $[\ U' \mid A\ ]$ in which $U'$ is the reduced row-echelon form which is row equivalent to $U$:

$$[\ U \mid I_3\ ] = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -1 & 3 & -5 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 7 & -2 & 1 \end{array}\right] = [\ U' \mid A\ ]$$

in which $U' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 1 & 0 \\ 7 & -2 & 1 \end{bmatrix}$

The rank of $U'$ is 2.

Define $A_\natural = [\ 7 \quad 2 \quad -1\ ]$. We have $\mathcal{N}(A_\natural) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ -4 \\ 11 \end{bmatrix}$.

*We want to express* $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$ *as the null space of some appropriate matrix with three columns.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.

We apply successive row operations starting from $[\ U \mid I_3\ ]$, in such a way to obtain some matrix $[\ U' \mid A\ ]$ in which $U'$ is the reduced row-echelon form which is row equivalent to $U$:

$$[\ U \mid I_3\ ] = \left[\begin{array}{cccc|ccc} 0 & 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & -2 & 3 & -4 & 0 & 1 & 0 \\ 2 & 7 & -12 & 11 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 2 & -2 & -1 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 2 & 1 \end{array}\right] = [\ U' \mid A\ ]$$

in which $U' = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$

The rank of $U'$ is 2.

Define $A_\natural = [\ -3 \quad 2 \quad 1\ ]$. We have $\mathcal{N}(A_\natural) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$.

*We want to express* $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$ *as the null space of some appropriate matrix with three columns.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.

We apply successive row operations starting from $[\ U \mid I_3\ ]$, in such a way to obtain some matrix $[\ U' \mid A\ ]$ in which $U'$ is the reduced row-echelon form which is row equivalent to $U$:

$$[\ U \mid I_3\ ] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] = [\ U' \mid A\ ]$$

in which $U' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 3 & -2 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

The rank of $U'$ is 3. We have $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}) = \mathbb{R}^3 = \mathcal{N}(\mathcal{O}_{1\times 3})$.

(d) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

*We want to express* $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$ *as the null space of some appropriate matrix with four columns.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.

We apply successive row operations starting from $[\ U \mid I_4\ ]$, in such a way to obtain some matrix $[\ U' \mid A\ ]$ in which $U'$ is the reduced row-echelon form which is row equivalent to $U$:

$$[\ U \mid I_3\ ] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 4 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -7 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right] = [\ U' \mid A\ ]$$

in which $U' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 4 & 0 & -1 & 0 \\ -7 & 1 & 2 & 0 \\ 4 & -1 & -1 & 0 \\ 2 & -1 & -1 & 1 \end{bmatrix}$

The rank of $U'$ is 3.

Define $A_\natural = [\ 2 \quad -1 \quad -1 \quad 1\ ]$. We have $\mathcal{N}(A_\natural) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.

(e) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix}$.

*We want to express* $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\})$ *as the null space of some appropriate matrix with four columns.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ]$.

We apply successive row operations starting from $[\ U \mid I_4\ ]$, in such a way to obtain some matrix $[\ U' \mid A\ ]$ in which $U'$ is the reduced row-echelon form which is row equivalent to $U$:

$$\begin{aligned} [\ U \mid I_3\ ] &= \left[\begin{array}{ccccc|cccc} 1 & 2 & 7 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 5 & -1 & 9 & 0 & 0 & 1 & 0 \\ 1 & -1 & -5 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \\ &\longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{ccccc|cccc} 1 & 0 & -1 & 0 & 3 & -3 & 5 & 0 & -1 \\ 0 & 1 & 4 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 9 & -16 & 1 & 4 \end{array}\right] = [\ U' \mid A\ ] \end{aligned}$$

in which $U' = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -3 & 5 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 9 & -16 & 1 & 4 \end{bmatrix}$

The rank of $U'$ is 3.

Define $A_\natural = \begin{bmatrix} 9 & -16 & 1 & 4 \end{bmatrix}$. We have $\mathcal{N}(A_\natural) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\})$.

(f) *Let* $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

*We want to express* $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$ *as the null space of some appropriate matrix with five columns.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.

We apply successive row operations starting from $[\ U \mid I_5\ ]$, in such a way to obtain some matrix $[\ U' \mid A\ ]$ in which $U'$ is the reduced row-echelon form which is row equivalent to $U$:

$$[\ U \mid I_5\ ] = \left[\begin{array}{ccc|ccccc} -2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & -4 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & 4 \end{array}\right] = [\ U' \mid A\ ]$$

in which $U' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \end{bmatrix}$

The rank of $U'$ is 3.

Define $A_\natural = \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \end{bmatrix}$. We have $\mathcal{N}(A_\natural) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$.

(g) *Let* $\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

*We want to express* $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\})$ *as the null space of some appropriate matrix with seven columns.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.

We apply successive row operations starting from $[\ U \mid I_7\ ]$, in such a way to obtain some matrix $[\ U' \mid A\ ]$ in which $U'$ is the reduced row-echelon form which is row equivalent to $U$:

$$[\ U \mid I_7\ ] = \left[\begin{array}{cccc|ccccccc} -4 & -2 & -1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 6 & -6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cccc|ccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -6 & 6 \end{array}\right] = [\ U' \mid A\ ]$$

in which $U' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \end{bmatrix}$

The rank of $U'$ is 4.

Define $A_\natural = \begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \end{bmatrix}$. We have $\mathcal{N}(A_\natural) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.