1. Recall an observation from the handout *Homogeneous systems and null spaces*:

   Suppose we are given an $(m \times n)$ matrix B.
   To determine $\mathcal{N}(B)$ is the same as giving an 'explicit' description of the solution set of the homogeneous system $\mathcal{LS}(B, \mathbf{0})$ through set language, in terms of (hopefully just a few) solutions of the system. That amounts to finding all solutions of $\mathcal{LS}(B, \mathbf{0})$.

   In practice, this is what we proceed with the above:

   Suppose B' is the reduced row-echelon form which is row-equivalent to B.
   Suppose the rank of B' is r. Write $k = n - r$.
   When $k = 0$, $\mathcal{N}(B) = \{0\}$.
   Suppose $k > 0$. Then those (few) solutions $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ of $\mathcal{LS}(B, \mathbf{0})$ needed for expressing all solutions of $\mathcal{LS}(B, \mathbf{0})$ are 'read off' as solutions of $\mathcal{LS}(B', \mathbf{0})$ for which one free variable takes the value 1 and all other free variable take the value 0.
   In conclusion we have

$$\begin{aligned}\mathcal{N}(B) &= \mathcal{N}(B') \\ &= \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid c_1, c_2, \cdots, c_k \in \mathbb{R}\} = \mathsf{Span}\left(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}\right).\end{aligned}$$

   A natural follow-up question is: *can this process be reversed?*
   (And in what sense can this be reversed?)

2. **Question.**

Suppose we are given a collection of vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ in $\mathbb{R}^n$.

Can we express

$$\mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}\right)$$

as the null space of some appropriate matrix with n columns?

Answer.

The answer is '*yes*', and will be provided by Theorem (M).

Remark.

Hence, the null space of a matrix is the span of some vectors, while the span of several vectors is the null space of some matrix.

The notions of *null space*, *span*, *column space* are manifestations of the same mathematical concept.

3. **Theorem (M).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^n$, *and* $U = \left[\ \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_q\ \right]$.

Denote by U' *the reduced row-echelon form which is row-equivalent to* U.

Denote the rank of U' *by* r, *and suppose* $0 < r < q$. *Write* $p = n - r$.

Suppose A *is a non-singular and invertible* $(n \times n)$*-matrix which satisfies* $U' = AU$.

Denote by $A_\natural$ *the* $(p \times n)$*-matrix constituted by the bottom* p *rows of* A.

Then $\mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}\right) = \mathcal{C}(U) = \mathcal{N}\left(A_\natural\right)$.

Remarks on the statement of Theorem (M).

(a) Theorem (M) is meaningful (and useful) because of the validity of the result $(\star)$ below from the handout *Row equivalence in terms of multiplication by non-singular and invertible matrices*:

$(\star)$ *Let* C, D *be* $(n \times q)$*-matrices.*

The statements below are logically equivalent:

i. C *is row-equivalent to* D.

ii. *There exists some non-singular and invertible* $(n \times n)$*-square matrix* A *such that* $D = AC$.

(b) Theorem (M) is formulated in such a way to avoid the complications in having to cover the 'extreme cases' '$r = 0$', '$r = n$' within the statement.

i. When $r = 0$, we have $U = \mathcal{O}_{n\times q}$ and $\mathcal{C}(U) = \{\mathbf{0}_n\} = \mathcal{N}(I_n)$.

ii. When $r = n$, we have $\mathcal{C}(U) = \mathcal{C}(I_n) = \mathbb{R}^n = \mathcal{N}(\mathcal{O}_{1\times n})$.

4. **Proof of Theorem (M).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^n$, and $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_q \,\right]$. We have $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \mathcal{C}(U)$.

Denote by U' the reduced row-echelon form which is row-equivalent to U.

Denote the rank of U' by r, and suppose $0 < r < q$. Write $p = n - r$.

Suppose A is a non-singular and invertible $(n \times n)$-matrix which satisfies $U' = AU$.

Denote by $A_{\natural}$ the $(p \times n)$-matrix constituted by the bottom p rows of A.

Denote by $A_{\sharp}$ the $(r \times n)$-matrix constituted by the top r rows of A.

So

$$A = \left[\begin{array}{c} A_{\sharp} \\ \hline A_{\natural} \end{array}\right].$$

Denote by $U'_{\sharp}$ the $(r \times q)$-matrix constituted by the top r rows of U'.

So

$$U' = \left[\begin{array}{c} U'_{\sharp} \\ \hline \mathcal{O}_{p \times q} \end{array}\right].$$

We want to verify that $\mathcal{C}(U) = \mathcal{N}(A_{\natural})$.

- [We verify that every vector in $\mathcal{C}(U)$ belongs to $\mathcal{N}\left(A_\natural\right)$.

  This amounts to verify the statement '*For any* $\mathbf{t} \in \mathbb{R}^n$, *if* $\mathbf{t} \in \mathcal{C}(U)$ *then* $\mathbf{t} \in \mathcal{N}\left(A_\natural\right)$'.]

  Pick any $\mathbf{t} \in \mathbb{R}^n$.
  Suppose $\mathbf{t} \in \mathcal{C}(U)$. Then there exists some $\mathbf{z} \in \mathbb{R}^q$ such that $\mathbf{t} = U\mathbf{z}$.
  We have

$$\left[\begin{array}{c} U'_\sharp\mathbf{z} \\ \hline \mathbf{0}_p \end{array}\right] = \left[\begin{array}{c} U'_\sharp\mathbf{z} \\ \hline \mathcal{O}_{p\times q}\mathbf{z} \end{array}\right] = \left[\begin{array}{c} U'_\sharp \\ \hline \mathcal{O}_{p\times q} \end{array}\right]\mathbf{z} = U'\mathbf{z} = AU\mathbf{z} = A\mathbf{t} = \left[\begin{array}{c} A_\sharp \\ \hline A_\natural \end{array}\right]\mathbf{t} = \left[\begin{array}{c} A_\sharp\mathbf{t} \\ \hline A_\natural\mathbf{t} \end{array}\right].$$

  Then $A_\natural\mathbf{t} = \mathbf{0}_p$.
  Therefore $\mathbf{t} \in \mathcal{N}(A_\natural)$.

- [We verify that every vector in $\mathcal{N}\left(A_\natural\right)$ belongs to $\mathcal{C}(U)$.

  This amounts to verify the statement '*For any* $\mathbf{t} \in \mathbb{R}^n$, *if* $\mathbf{t} \in \mathcal{N}\left(A_\natural\right)$ *then* $\mathbf{t} \in \mathcal{C}(U)$'.]

  Pick any $\mathbf{t} \in \mathbb{R}^n$.
  Suppose $\mathbf{t} \in \mathcal{N}(A_\natural)$. Then $A_\natural \mathbf{t} = \mathbf{0}_p$.
  We have $A\mathbf{t} = \left[\begin{array}{c} A_\sharp \\ \hline A_\natural \end{array}\right]\mathbf{t} = \left[\begin{array}{c} A_\sharp \mathbf{t} \\ \hline A_\natural \mathbf{t} \end{array}\right] = \left[\begin{array}{c} A_\sharp \mathbf{t} \\ \hline \mathbf{0}_p \end{array}\right]$.
  Consider the system $\mathcal{LS}(U, \mathbf{t})$. Its augmented matrix representation is $\left[\, U \,\middle|\, \mathbf{t} \,\right]$,
  Since A is non-singular, $\left[\, U \,\middle|\, \mathbf{t} \,\right]$ is row-equivalent to the matrix $A\left[\, U \,\middle|\, \mathbf{t} \,\right]$, which is explicitly given by

$$A\left[\, U \,\middle|\, \mathbf{t} \,\right] = \left[\, U' \,\middle|\, A\mathbf{t} \,\right] = \left[\begin{array}{c|c} U'_\sharp & A_\sharp \mathbf{t} \\ \hline \mathcal{O}_{p\times q} & A_\natural \mathbf{t} \end{array}\right] = \left[\begin{array}{c|c} U'_\sharp & A_\sharp \mathbf{t} \\ \hline \mathcal{O}_{p\times q} & \mathbf{0}_p \end{array}\right],$$

  which is a reduced row-echelon form whose last column is not a pivot column.
  Then the system $\mathcal{LS}(U, \mathbf{t})$ is consistent.
  Therefore there exists some $\mathbf{z} \in \mathbb{R}^q$ such that $U\mathbf{z} = \mathbf{t}$.
  Hence $\mathbf{t} \in \mathcal{C}(U)$.

It follows that $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}) = \mathcal{C}(U) = \mathcal{N}\left(A_\natural\right)$.

5. Theorem (M) suggests an 'algorithm' with which we can express the span of some 'concretely' given vectors in $\mathbb{R}^n$ explicitly as the null space of a 'concretely' determined matrix with n columns.

'Algorithm' associated with Theorem (M).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q \in \mathbb{R}^n$. We are going to write down a matrix with n columns whose null space is the same as the span of these vectors.

- **Step (0).**
  If $\mathbf{u}_1 = \mathbf{u}_2 = \cdots = \mathbf{u}_q = \mathbf{0}_n$ then $\mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\}\right) = \mathcal{N}(I_n)$.
  From now on assume $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are not all zero vectors.
- **Step (1).**
  Form the matrix $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n \,\right]$.
  Further form the matrix $\left[\, U \,\middle|\, I_n \,\right]$.
- **Step (2).**
  Apply row operations on $\left[\, U \,\middle|\, I_n \,\right]$ so as to result in the matrix $\left[\, U' \,\middle|\, A \,\right]$, which is row-equivalent to $\left[\, U \,\middle|\, I_n \,\right]$, and in which U' is the reduced row-echelon form row-equivalent to U.
- **Step (3).**
  Inspect the matrix U'. Denote its rank by r.
  - $*$ Suppose $r = n$. Then $\mathsf{Span}\left(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\right) = \mathcal{N}(\mathcal{O}_{1\times n})$.
  - $*$ Suppose $r < n$. Write $p = n - r$. Denote by $A_\natural$ the $(p \times n)$-matrix given by the bottom p rows of A.
    Then $\mathsf{Span}\left(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\right) = \mathcal{N}\left(A_\natural\right)$.

6. **Illustrations.**

(a) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$.

We want to express $\mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}\right)$ *as the null space of some appropriate matrix with three columns.*

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\right]$.

We apply successive row operations starting from $\left[\, U \,\middle|\, I_3 \,\right]$, in such a way to obtain some matrix $\left[\, U' \,\middle|\, A \,\right]$ in which U' is the reduced row-echelon form row-equivalent to U:

$$\left[\, U \,\middle|\, I_3 \,\right] = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -1 & 3 & -5 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 7 & -2 & 1 \end{array}\right] = \left[\, U' \,\middle|\, A \,\right]$$

in which $U' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 1 & 0 \\ 7 & -2 & 1 \end{bmatrix}$

The rank of U' is 2.

Define $A_\natural = \begin{bmatrix} 7 & 2 & -1 \end{bmatrix}$. We have $\mathcal{N}\left(A_\natural\right) = \mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}\right)$.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ -4 \\ 11 \end{bmatrix}$.

We want to express $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$ *as the null space of some appropriate matrix with three columns.*

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\middle|\, \mathbf{u}_4 \,\right]$.

We apply successive row operations starting from $\left[\, U \,\middle|\, I_3 \,\right]$, in such a way to obtain some matrix $\left[\, U' \,\middle|\, A \,\right]$ in which U' is the reduced row-echelon form row-equivalent to U:

$$\left[\, U \,\middle|\, I_3 \,\right] = \left[\begin{array}{cccc|ccc} 0 & 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & -2 & 3 & -4 & 0 & 1 & 0 \\ 2 & 7 & -12 & 11 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 2 & -2 & -1 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 2 & 1 \end{array}\right] = \left[\, U' \,\middle|\, A \,\right]$$

in which $U' = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$

The rank of U' is 2.

Define $A_\natural = \begin{bmatrix} -3 & 2 & 1 \end{bmatrix}$. We have $\mathcal{N}\left(A_\natural\right) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$.

We want to express $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\})$ *as the null space of some appropriate matrix with three columns.*

Define $U = \left[\ \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3\ \right]$.

We apply successive row operations starting from $\left[\ U \,\middle|\, I_3\ \right]$, in such a way to obtain some matrix $\left[\ U' \,\middle|\, A\ \right]$ in which U' is the reduced row-echelon form row-equivalent to U:

$$\left[\ U \,\middle|\, I_3\ \right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] = \left[\ U' \,\middle|\, A\ \right]$$

in which $U' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 3 & -2 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

The rank of U' is 3. We have $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}) = \mathbb{R}^3 = \mathcal{N}(\mathcal{O}_{1\times 3})$.

(d) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

We want to express $\mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}\right)$ *as the null space of some appropriate matrix with four columns.*

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\middle|\, \mathbf{u}_4 \,\right]$.

We apply successive row operations starting from $\left[\, U \,\middle|\, I_4 \,\right]$, in such a way to obtain some matrix $\left[\, U' \,\middle|\, A \,\right]$ in which U' is the reduced row-echelon form row-equivalent to U:

$$\left[\, U \,\middle|\, I_3 \,\right] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 4 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -7 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right] = \left[\, U' \,\middle|\, A \,\right]$$

in which $U' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 4 & 0 & -1 & 0 \\ -7 & 1 & 2 & 0 \\ 4 & -1 & -1 & 0 \\ 2 & -1 & -1 & 1 \end{bmatrix}$

The rank of U' is 3.

Define $A_\natural = \left[\, 2 \;\; -1 \;\; -1 \;\; 1 \,\right]$. We have $\mathcal{N}\left(A_\natural\right) = \mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}\right)$.

(e) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix}$.

We want to express $\mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}\right)$ *as the null space of some appropriate matrix with four columns.*

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\middle|\, \mathbf{u}_4 \,\middle|\, \mathbf{u}_5 \,\right]$.

We apply successive row operations starting from $\left[\, U \,\middle|\, I_4 \,\right]$, in such a way to obtain some matrix $\left[\, U' \,\middle|\, A \,\right]$ in which U' is the reduced row-echelon form row-equivalent to U:

$$\left[\, U \,\middle|\, I_3 \,\right] = \left[\begin{array}{ccccc|cccc} 1 & 2 & 7 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 5 & -1 & 9 & 0 & 0 & 1 & 0 \\ 1 & -1 & -5 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccccc|cccc} 1 & 0 & -1 & 0 & 3 & -3 & 5 & 0 & -1 \\ 0 & 1 & 4 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 9 & -16 & 1 & 4 \end{array}\right] = \left[\, U' \,\middle|\, A \,\right]$$

in which $U' = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -3 & 5 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 9 & -16 & 1 & 4 \end{bmatrix}$

The rank of U' is 3.

Define $A_\natural = \begin{bmatrix} 9 & -16 & 1 & 4 \end{bmatrix}$. We have $\mathcal{N}\left(A_\natural\right) = \mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}\right)$.

(f) *Let* $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

We want to express $\mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}\right)$ *as the null space of some appropriate matrix with five columns.*

Define $U = \left[\ \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3\ \right]$.

We apply successive row operations starting from $\left[\ U \,\middle|\, I_5\ \right]$, in such a way to obtain some matrix $\left[\ U' \,\middle|\, A\ \right]$ in which U' is the reduced row-echelon form which is row equivalent to U:

$$\left[\ U \,\middle|\, I_5\ \right] = \left[\begin{array}{ccc|ccccc} -2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & -4 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & 4 \end{array}\right] = \left[\ U' \,\middle|\, A\ \right]$$

in which $U' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \end{bmatrix}$

The rank of U' is 3. Define $A_\natural = \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \end{bmatrix}$. We have $\mathcal{N}\left(A_\natural\right) = \mathsf{Span}\left(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}\right)$.

(g) *Let* $\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

We want to express Span $(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\})$ *as the null space of some appropriate matrix with seven columns.*

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\middle|\, \mathbf{u}_4 \,\right]$.

We apply successive row operations starting from $\left[\, U \,\middle|\, I_7 \,\right]$, in such a way to obtain some matrix $\left[\, U' \,\middle|\, A \,\right]$ in which U' is the reduced row-echelon form which is row equivalent to U:

$$\left[\, U \,\middle|\, I_7 \,\right] = \left[\begin{array}{cccc|ccccccc} -4 & -2 & -1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 6 & -6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cccc|ccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -6 & 6 \end{array}\right] = \left[\, U' \,\middle|\, A \,\right]$$

in which $U' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \end{bmatrix}$

The rank of U' is 4. Define $A_\natural = \begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \end{bmatrix}$. We have $\mathcal{N}(A_\natural) = $ Span $(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.