

1. Recall Theorem (B) from the handout *Linear Combinations*:

Let  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n$  be vectors in  $\mathbb{R}^m$ .

Every linear combination of (finitely many) linear combinations of  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n$  is a linear combination of  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n$ .

Also recall the definition for the notion of *span*:

Let  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$  be ('finitely many') vectors in  $\mathbb{R}^m$ .

The span of (the set of vectors)  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$  is defined to be the set

$$\{ \mathbf{y} \in \mathbb{R}^m : \mathbf{y} \text{ is a linear combination of } \mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n \}$$

We denote this set by  $\text{Span}(\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n\})$ .

2. In this handout we need to handle 'equality questions' on sets. We introduce the definition for the notion of *set equality* (or recall it from the handout *The use of set notations in linear algebra*):

Let  $K, L$  be sets (of vectors in  $\mathbb{R}^n$ ).

We say that  $K, L$  are equal to each other, and write  $K = L$  if and only if both of  $(\dagger)$ ,  $(\ddagger)$  are true:

$(\dagger)$  For any  $\mathbf{u}$ , if  $\mathbf{u} \in K$  then  $\mathbf{u} \in L$ .

$(\ddagger)$  For any  $\mathbf{v}$ , if  $\mathbf{v} \in L$  then  $\mathbf{v} \in K$ .

### 3. Theorem (1).

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^m$ .

Suppose each of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  is a linear combination of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .

Further suppose each of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .

Then  $\text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .

#### Remark.

The equality ' $\text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ ' is a set equality. What such an equality means is that the statements (†), (‡) below hold simultaneously:

(†) For any  $\mathbf{y} \in \mathbb{R}^m$ , if  $\mathbf{y} \in \text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\})$  then  $\mathbf{y} \in \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .

(‡) For any  $\mathbf{y} \in \mathbb{R}^m$ , if  $\mathbf{y} \in \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$  then  $\mathbf{y} \in \text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\})$ .

#### 4. Proof of Theorem (1).

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^m$ .

Suppose each of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  is a linear combination of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .

Further suppose each of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .

We deduce the statement ( $\dagger$ ):

( $\dagger$ ) ‘For any  $\mathbf{y} \in \mathbb{R}^m$ , if  $\mathbf{y} \in \text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\})$  then  $\mathbf{y} \in \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .’

- Pick any  $\mathbf{y} \in \mathbb{R}^m$ . Suppose  $\mathbf{y} \in \text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\})$ .

[Reminder: We want to see why  $\mathbf{y}$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .]

By definition,  $\mathbf{y}$  is a linear combination of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .

Each of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .

Then, by Theorem (B),  $\mathbf{y}$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .

Therefore  $\mathbf{y} \in \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .

Modifying the above argument for ( $\dagger$ ), we also deduce the statement ( $\ddagger$ ):

( $\ddagger$ ) ‘For any  $\mathbf{y} \in \mathbb{R}^m$ , if  $\mathbf{y} \in \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$  then  $\mathbf{y} \in \text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\})$ .’

It follows that  $\text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .

5. The converse of Theorem (1) is an immediate consequence of Lemma (2).

**Lemma (2).**

*Suppose  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$  are vectors in  $\mathbb{R}^m$ .*

*Then each of  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$  belongs to  $\text{Span}(\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n\})$ .*

**Theorem (3). (Converse of Theorem (1).)**

*Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^m$ .*

*Suppose  $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .*

*Then each of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  is a linear combination of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .*

*Also each of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .*

6. We may combine Theorem (1) and Theorem (3) to obtain Theorem (K):

**Theorem (K).**

*Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^m$ . The statements below are logically equivalent:*

- (a) *Each of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  is a linear combination of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ , and each of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .*
- (b)  $\text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .

7. **Corollary to Theorem (K).**

*Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_p$  be vectors in  $\mathbb{R}^m$ . The statements below are logically equivalent:*

- (a) *Each of  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_p$  is a linear combination of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .*
- (b)  $\text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_p\}) = \text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\})$ .

## 8. Illustrations of Theorem (K).

$$(a) \text{ Span } \left( \left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \right) = \text{Span } \left( \left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\} \right).$$

Reason: Each of  $\begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  is a linear combination of  $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ .

Below is the detail:

$$\begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

$$(b) \text{ Span } \left( \left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\} \right) = \text{Span } \left( \left\{ \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \right).$$

Reason: Each of  $\begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  is a linear combination of  $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ .

Below is the detail:

$$\begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

Moreover, each of  $\begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  is a linear combination of

$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ . Below is the detail:

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

### 9. **Lemma (4).**

Let  $H$  be an  $(m \times n)$ -matrix, and  $G$  be an  $(m \times k)$ -matrix.

Suppose  $A$  is a  $(m \times m)$ -square matrix, and  $\mathcal{C}(H) = \mathcal{C}(G)$ .

Then  $\mathcal{C}(AH) = \mathcal{C}(AG)$ .

### 10. **Proof of Lemma (4).**

Let  $H$  be an  $(m \times n)$ -matrix, and  $G$  be an  $(m \times k)$ -matrix.

Suppose  $A$  is a  $(m \times m)$ -square matrix, and  $\mathcal{C}(H) = \mathcal{C}(G)$ .

[We are going to verify the set equality  $\mathcal{C}(AH) = \mathcal{C}(AG)$ . This amounts to deducing (with the assumption stated earlier) that both  $(\dagger)$ ,  $(\ddagger)$  are true:

$(\dagger)$  For any  $\mathbf{y} \in \mathbb{R}^m$ , if  $\mathbf{y} \in \mathcal{C}(AG)$  then  $\mathbf{y} \in \mathcal{C}(AH)$ .

$(\ddagger)$  For any  $\mathbf{y} \in \mathbb{R}^m$ , if  $\mathbf{y} \in \mathcal{C}(AH)$  then  $\mathbf{y} \in \mathcal{C}(AG)$ .

We will use the arguments in two separate passages, one for each of  $(\dagger)$ ,  $(\ddagger)$ .]



- [Here we verify (†).]

Suppose  $\mathbf{y} \in \mathcal{C}(AH)$ . Then, by the definition of  $\mathcal{C}(AH)$ , there exist some  $\mathbf{x} \in \mathbb{R}^n$  such that  $\mathbf{y} = AH\mathbf{x}$ .

[Reminder: We want to deduce  $\mathbf{y} \in \mathcal{C}(AG)$ . So we ask whether we can conceive some appropriate  $\mathbf{w} \in \mathbb{R}^k$  which satisfies  $\mathbf{y} = (AG)\mathbf{w}$ .

How to conceive such a  $\mathbf{w}$ ? Compare the equality ' $\mathbf{y} = AH\mathbf{x}$ ' which we have already known to be true, with the desired equality ' $\mathbf{y} = AG\mathbf{w}$ ', which we hope to be true.

This suggests we ask if there is indeed some  $\mathbf{w} \in \mathbb{R}^k$  which satisfies  $H\mathbf{x} = G\mathbf{w}$ . It turns out that the answer is *yes*.]

By the definition of  $\mathcal{C}(H)$ ,  $H\mathbf{x} \in \mathcal{C}(H)$ .

Then by assumption  $H\mathbf{x} \in \mathcal{C}(G)$ .

Then, by the definition of  $\mathcal{C}(G)$ , there exists some  $\mathbf{w} \in \mathbb{R}^k$  such that  $H\mathbf{x} = G\mathbf{w}$ .

Now  $\mathbf{y} = AH\mathbf{x} = AG\mathbf{w}$ .

Then, by the definition of  $\mathcal{C}(AG)$ , we have  $\mathbf{y} \in \mathcal{C}(AG)$ .

- By modifying the above argument (through changing the symbols appropriately), we also deduce that for any  $\mathbf{y} \in \mathbb{R}^m$ , if  $\mathbf{y} \in \mathcal{C}(AG)$  then  $\mathbf{y} \in \mathcal{C}(AH)$ .

11. **Lemma (5).** (A ‘partial converse’ of Lemma (5).)

Let  $H$  be an  $(m \times n)$ -matrix, and  $G$  be an  $(m \times k)$ -matrix.

Suppose  $A$  is a non-singular  $(m \times m)$ -square matrix, and  $\mathcal{C}(AH) = \mathcal{C}(AG)$ .

Then  $\mathcal{C}(H) = \mathcal{C}(G)$ .

12. **Proof of Lemma (5).**

[We are going to make a clever application of Lemma (4) so that we don’t have to prove a set equality with direct reference to the definition of set equalities.]

Let  $H$  be an  $(m \times n)$ -matrix, and  $G$  be an  $(m \times k)$ -matrix.

Suppose  $A$  is a non-singular  $(m \times m)$ -square matrix, and  $\mathcal{C}(AH) = \mathcal{C}(AG)$ .

By assumption,  $A$  has a matrix inverse, namely the  $(m \times m)$ -square matrix  $A^{-1}$ .

We have  $H = A^{-1}(AH)$ .

Then  $\mathcal{C}(H) = \mathcal{C}(A^{-1}(AH))$

We also have  $G = A^{-1}(AG)$ .

Then  $\mathcal{C}(G) = \mathcal{C}(A^{-1}(AG))$ .

By assumption,  $\mathcal{C}(AH) = \mathcal{C}(AG)$ . Then, by Lemma (4), we have  $\mathcal{C}(A^{-1}(AH)) = \mathcal{C}(A^{-1}(AG))$ .

Therefore  $\mathcal{C}(H) = \mathcal{C}(A^{-1}(AH)) = \mathcal{C}(A^{-1}(AG)) = \mathcal{C}(G)$ .

13. We combine Lemma (4) and Lemma (5) to obtain Theorem (L) below:

**Theorem (L).**

*Let  $H$  be an  $(m \times n)$ -matrix, and  $G$  be an  $(m \times k)$ -matrix.*

*Suppose  $A$  is a non-singular  $(m \times m)$ -square matrix.*

*Then the statements below are logically equivalent:*

(a)  $\mathcal{C}(H) = \mathcal{C}(G)$ .

(b)  $\mathcal{C}(AH) = \mathcal{C}(AG)$ .

**Remark.**

In plain words, this result is saying that

*the equality between column spaces of matrices (though not necessarily the individual matrices themselves) are preserved upon multiplication of the same non-singular square matrix from the left to the matrices.*

When we think in terms of row operations, this result is saying that

*the equality between column spaces of matrices (though not necessarily the individual matrices themselves) are preserved upon the application of the same sequence of row operations to the matrices.*

14. Under the ‘dictionary’ between the notion of *column space* and *span*, Lemma (4), Lemma (5) and Theorem (L) respectively translate into Lemma (4’), Lemma (5’) and Theorem (L’) below.

**Lemma (4’).**

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^m$ .

Suppose  $A$  is a  $(m \times m)$ -square matrix, and

$$\text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}).$$

$$\text{Then Span} (\{A\mathbf{u}_1, A\mathbf{u}_2, \dots, A\mathbf{u}_n\}) = \text{Span} (\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_k\}).$$

**Lemma (5’). (A ‘partial converse’ of Lemma (4’).)**

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^m$ .

Suppose  $A$  is a non-singular  $(m \times m)$ -square matrix, and

$$\text{Span} (\{A\mathbf{u}_1, A\mathbf{u}_2, \dots, A\mathbf{u}_n\}) = \text{Span} (\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_k\}).$$

$$\text{Then Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}).$$

**Theorem (L'). (Re-formulation of Theorem (L) under the 'dictionary' between span and column space.)**

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^m$ .

Suppose  $A$  is a non-singular  $(m \times m)$ -square matrix.

Then the statements below are logically equivalent:

- (a)  $\text{Span} (\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span} (\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .
- (b)  $\text{Span} (\{A\mathbf{u}_1, A\mathbf{u}_2, \dots, A\mathbf{u}_n\}) = \text{Span} (\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_k\})$ .

**Remark.**

In plain words, this result is saying that

*the equality between spans of vectors (though not necessarily the individual vectors themselves) are preserved upon multiplication of the same non-singular square matrix from the left to the vectors.*

When we think in terms of row operations, this result is saying that

*the equality between spans of vectors (though not necessarily the individual vectors themselves) are preserved upon the application of the same sequence of row operations to the vectors.*

15. **Theorem (6).** (Generalization of Lemma (4) and Lemma (5).)

Let  $H$  be an  $(m \times n)$ -matrix, and  $G$  be an  $(m \times k)$ -matrix.

Let  $A$  be a  $(p \times m)$ -matrix.

(a) Suppose  $\mathcal{C}(H) = \mathcal{C}(G)$ .

Then  $\mathcal{C}(AH) = \mathcal{C}(AG)$ .

(b) Suppose  $\mathcal{N}(A) = \{\mathbf{0}\}$ , and  $\mathcal{C}(AH) = \mathcal{C}(AG)$ .

Then  $\mathcal{C}(AH) = \mathcal{C}(AG)$ .

**Theorem (6').** (Generalization of Lemma (4') and Lemma (5').)

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^m$ . Let  $A$  be a  $(p \times m)$ -matrix.

(a) Suppose  $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .

Then  $\text{Span}(\{A\mathbf{u}_1, A\mathbf{u}_2, \dots, A\mathbf{u}_n\}) = \text{Span}(\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_k\})$ .

(b) Suppose  $\mathcal{N}(A) = \{\mathbf{0}\}$ , and  $\text{Span}(\{A\mathbf{u}_1, A\mathbf{u}_2, \dots, A\mathbf{u}_n\}) = \text{Span}(\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_k\})$ .

Then  $\text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}) = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\})$ .