MATH1030 Further examples on linear dependence and linear independence.

1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^m$. Prove that the vectors $\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}$ are linearly dependent.

2. Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^m$. Prove that the statements below are logically equivalent:

   (1) $\mathbf{u}, \mathbf{v}$ *are linearly dependent.*

   (2) *One of* $\mathbf{u}, \mathbf{v}$ *is a scalar multiple of the other.*

3. Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^m$. Prove that the statements below are logically equivalent:

   (1) *The vectors* $\mathbf{u}, \mathbf{v}$ *are linearly independent.*

   (2) *The vectors* $\mathbf{u}, -\mathbf{v}$ *are linearly independent.*

   (3) *The vectors* $\mathbf{u} + \mathbf{v}, \mathbf{u}$ *are linearly independent.*

   (4) *The vectors* $\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}$ *are linearly independent.*

4. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^m$. Prove that the statements below are logically equivalent:

   (1) *The vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are linearly independent.*

   (2) *The vectors* $\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}$ *are linearly independent.*

   (3) *The vectors* $\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{w} + \mathbf{u}$ *are linearly independent.*

5. Let $\mathbf{t}_1, \mathbf{t}_2$ be vectors in $\mathbb{R}^m$.
   Let A be a (2×2)-square matrix, whose (i, j)-th entry is a_{ij}.
   Define $\mathbf{u}_1 = a_{11}\mathbf{t}_1 + a_{21}\mathbf{t}_2$ and $\mathbf{u}_2 = a_{12}\mathbf{t}_1 + a_{22}\mathbf{t}_2$.
   Suppose $\mathbf{t}_1, \mathbf{t}_2$ are linearly independent.
   Prove that the statements below are logically equivalent:

   (1) *The vectors* $\mathbf{u}_1, \mathbf{u}_2$ *are linearly independent.*

   (2) *The matrix* A *is non-singular.*

6. Let $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4$ be vectors in $\mathbb{R}^m$.
   Let A be a (4×4)-square matrix, whose (i, j)-th entry is a_{ij}.
   For each $j = 1, 2, 3, 4$, define $\mathbf{u}_j = a_{1j}\mathbf{t}_1 + a_{2j}\mathbf{t}_2 + a_{3j}\mathbf{t}_3 + a_{4j}\mathbf{t}_4$.
   Suppose $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4$ are linearly independent.
   Prove that the statements below are logically equivalent:

   (1) *The vectors* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *are linearly independent.*

   (2) *The matrix* A *is non-singular.*

7. Let $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ be vectors in $\mathbb{R}^m$.
   Let A be a (5×3)-square matrix, whose (i, j)-th entry is a_{ij}.
   For each $j = 1, 2, 3$, define $\mathbf{u}_j = a_{1j}\mathbf{t}_1 + a_{2j}\mathbf{t}_2 + a_{3j}\mathbf{t}_3 + a_{4j}\mathbf{t}_4 + a_{5j}\mathbf{t}_5$.
   Suppose $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5$ are linearly independent.
   Prove that the statements below are logically equivalent:

   (1) *The vectors* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linearly independent.*

   (2) $\mathcal{N}(A) = \{\mathbf{0}_3\}$.