MATH1030 Further examples on linear combinations.

1. Consider the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$ and $\mathbf{v}$ below. Determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$. Where it is, also express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$.

   (a) $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}2\\-1\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}1\\-2\\5\end{bmatrix}$.

   (b) $\mathbf{u}_1 = \begin{bmatrix}1\\-3\\2\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\-4\\-1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\-5\\7\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}2\\-5\\3\end{bmatrix}$.

   (c) $\mathbf{u}_1 = \begin{bmatrix}1\\2\\0\\3\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}2\\-1\\2\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}3\\9\\-4\\-2\end{bmatrix}$.

   (d) $\mathbf{u}_1 = \begin{bmatrix}2\\4\\6\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}6\\10\\20\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}4\\13\\19\end{bmatrix}$.

   (e) $\mathbf{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-2\\3\\9\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}3\\1\\3\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\-2\\4\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}2\\1\\4\end{bmatrix}$.

2. Verify the statements below:

   (a) Every vector in $\mathbb{R}^4$ is a linear combination of $\mathbf{u}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$.

   (b) Every vector in $\mathbb{R}^4$ is a linear combination of $\mathbf{u}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

   (c) Every vector in $\mathbb{R}^4$ is a linear combination of $\mathbf{u}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$.

   (d) Every vector in $\mathbb{R}^4$ is a linear combination of $\mathbf{u}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$.

   (e) Every vector in $\mathbb{R}^4$ is a linear combination of $\mathbf{u}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}$.

3. (a) Let α be a real number, and let $\mathbf{u}_1 = \begin{bmatrix}3\\0\\-2\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\-1\\-5\end{bmatrix}$, $\mathbf{v}(\alpha) = \begin{bmatrix}1\\-2\\\alpha\end{bmatrix}$.

   For which value(s) of α is $\mathbf{v}(\alpha)$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2$? Justify your answer.

   (b) Let α be a real number, and let $\mathbf{u}_1 = \begin{bmatrix}1\\-3\\2\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\-1\\1\end{bmatrix}$, $\mathbf{v}(\alpha) = \begin{bmatrix}1\\\alpha\\5\end{bmatrix}$.

   For which value(s) of α is $\mathbf{v}(\alpha)$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2$? Justify your answer.

   (c) Let α be a real number, and let $\mathbf{u}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{u}_2(\alpha) = \begin{bmatrix}2\\3\\\alpha\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}3\\4\\5\end{bmatrix}$.

   For which value(s) of α is $\mathbf{v}$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2(\alpha)$? Justify your answer.

   (d) Let α be a real number, and let $\mathbf{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{u}_3(\alpha) = \begin{bmatrix}0\\1\\\alpha\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

   For which value(s) of α is $\mathbf{v}$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3(\alpha)$? Justify your answer.

Answer.

1. (a) Yes.
   $\mathbf{v} = -6\mathbf{u}_1 + 3\mathbf{u}_2 + 2\mathbf{u}_3$.
   (This is the only possible way to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

   (b) No.

   (c) Yes.
   $\mathbf{v} = \mathbf{u}_1 + 3\mathbf{u}_2 - 2\mathbf{u}_3$.
   (There are infinitely many possible ways to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

   (d) Yes.
   $\mathbf{v} = \frac{3}{2}\mathbf{u}_1 + 2\mathbf{u}_2 + \frac{1}{2}\mathbf{u}_3$.
   (There are infinitely many possible ways to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

   (e) Yes.
   $\mathbf{v} = \frac{17}{6}\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2 + 0 \cdot \mathbf{u}_3 + \frac{1}{2}\mathbf{u}_4$.
   (There are infinitely many possible ways to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.)

2. ——

3. (a) $\alpha = 8$ only.

   (b) $\alpha = 8$ only.

   (c) $\alpha = 4$ only.

   (d) α can be any real number not equal to 1.