MATH1030 Subspaces of $\mathbb{R}^n$ in contrast to general subsets of $\mathbb{R}^n$

1. Recall the definition for the notion of *subspaces of* $\mathbb{R}^n$:

   Let W be a set of vectors in $\mathbb{R}^n$.
   W is said to constitute a subspace of $\mathbb{R}^n$ if and only if the statements (S1), (S2), (S3) *hold:*

   (S1) $\mathbf{0}_n \in W$.

   (S2) *For any vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, if $\mathbf{u} \in W$ and $\mathbf{v} \in W$ then $\mathbf{u} + \mathbf{v} \in W$.*

   (S3) *For any vector $\mathbf{u}$ in $\mathbb{R}^n$, for any $\alpha \in \mathbb{R}$, if $\mathbf{u} \in W$ then $\alpha\mathbf{u} \in W$.*

   Also recall Theorem (E):

   Let W be a set of vectors in $\mathbb{R}^n$. Suppose W is a non-empty set of vectors.
   Then W is a subspace of $\mathbb{R}^n$ if and only if every linear combination of vectors in W belongs to W.

2. **Question.**

   Suppose we are given a set of vectors in $\mathbb{R}^n$, say, T. What do we mean when we say that T is not a subspace of $\mathbb{R}^n$?

   Answer.

   (a) According to Theorem (E), when the set T is non-empty, T will fail to be a subspace of $\mathbb{R}^n$ exactly when it happens that some linear combination of some vectors in T fails to 'stay in T'.
   When T is the empty set, T certainly fails to be a subspace of $\mathbb{R}^n$ according to definition.

   (b) More formally, according to logic and the definition to the notion of *subspace of* $\mathbb{R}^n$, such a set T would be a subspace of $\mathbb{R}^n$ exactly when all three statements (S1), (S2), (S3) would hold simultaneously:

   (S1) $\mathbf{0}_n \in T$.

   (S2) For any vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, if $\mathbf{u} \in T$ and $\mathbf{v} \in T$ then $\mathbf{u} + \mathbf{v} \in T$.

   (S3) For any vector $\mathbf{u}$ in $\mathbb{R}^n$, for any $\alpha \in \mathbb{R}$, if $\mathbf{u} \in T$ then $\alpha\mathbf{u} \in T$.

   Therefore, T fails to be a subspace of $\mathbb{R}^n$ exactly when at least one amongst the statements (S1), (S2), (S3) fails to hold.

   (c) In other words, T fails to be a subspace of $\mathbb{R}^n$ exactly when at least one amongst ($\sim$S1), ($\sim$S2), ($\sim$S3) holds:

   ($\sim$S1) $\mathbf{0}_n \notin T$.

   ($\sim$S2) There are some vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ such that $\mathbf{u} \in T$ and $\mathbf{v} \in T$ and $\mathbf{u} + \mathbf{v} \notin T$.

   ($\sim$S3) There are some vector $\mathbf{u}$ in $\mathbb{R}^n$ and some $\alpha \in \mathbb{R}$ such that $\mathbf{u} \in T$ and $\alpha\mathbf{u} \notin T$.

   The statements ($\sim$S1), ($\sim$S2), ($\sim$S3) are called the respective negations of the statements (S1), (S2), (S3).

3. **Further question.**

   Suppose we are given a set of vectors in $\mathbb{R}^n$, say, T. What shall we do when we guess that T is not a subspace of $\mathbb{R}^n$ and want to indeed verify that T is not a subspace of $\mathbb{R}^n$?

   Answer to further question.

   (a) It suffices to verify any one of ($\sim$S1), ($\sim$S2), ($\sim$S3).

   (b) If it is apparent to us which of ($\sim$S1), ($\sim$S2), ($\sim$S3) holds, we just proceed to verify it.

   (c) However, when it is not immediately clear which of ($\sim$S1), ($\sim$S2), ($\sim$S3) holds, we tend to proceed as described below (due to the relative 'complexities' in the logical structure of the statements):

   - **Step 1.**
     Check whether ($\sim$S1) holds. If *yes*, done.
     If *no*, proceed to Step 2.
   - **Step 2.**
     Check whether ($\sim$S3) holds. If *yes*, done.
     If *no*, proceed to Step 3.
   - **Step 3.**
     Check whether ($\sim$S2) holds. If *yes*, done.
     If *no*, go back to examine T again to see whether we wrongly guessed that T *was not a subspace of* $\mathbb{R}^n$.

4. **Non-examples of subspaces of $\mathbb{R}^2$, from sets of vectors in $\mathbb{R}^2$.**

In each of these examples, we can visualize the set of vectors concerned as a 'portion of the coordinate plane'. Through such a picture, we see immediately why the set of vectors concerned will fail to satisfy one of (S1), (S2), (S3).

(a) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \text{There exists some } s \in \mathbb{R} \text{ such that } \mathbf{x} = \begin{bmatrix} 1 \\ s \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. [Reminder: If W were a subspace or $\mathbb{R}^2$, then it would happen that $\mathbf{0} \in W$.]

Suppose $\mathbf{0} \in W$.

Then there would be some $s \in \mathbb{R}$ such that $\mathbf{0} = \begin{bmatrix} 1 \\ s \end{bmatrix}$.

Therefore $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 1 \\ s \end{bmatrix}$. Then $0 = 1$ (by comparison of the respective first entries).

Contradiction arises.

(b) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \text{There exists some } s \in \mathbb{R} \text{ such that } \mathbf{x} = \begin{bmatrix} s \\ 1-s \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. [Reminder: If W were a subspace or $\mathbb{R}^2$, then it would happen that $\mathbf{0} \in W$.]

Suppose $\mathbf{0} \in W$.

Then there would be some $s \in \mathbb{R}$ such that $\mathbf{0} = \begin{bmatrix} s \\ 1-s \end{bmatrix}$.

Therefore $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} = \begin{bmatrix} s \\ 1-s \end{bmatrix}$. Then $0 = s$ and $1 = s$ (by comparison of the respective entries). Hence $0 = s = 1$.

Contradiction arises.

(c) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \text{There exists some } s \in \mathbb{R} \text{ such that } s \geq 0 \text{ and } \mathbf{x} = \begin{bmatrix} s \\ s \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. [Reminder: If W were a subspace or $\mathbb{R}^2$, then whenever $\mathbf{x} \in W$ and $\alpha \in \mathbb{R}$, it would happen that $\alpha\mathbf{x} \in W$.]

Take $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We have $1 \geq 0$. Then $\mathbf{x} \in W$.

Take $\alpha = -1$. We have $\alpha \in \mathbb{R}$.

[Reminder: If W were a subspace of $\mathbb{R}^2$, then $\alpha\mathbf{x} \in W$.]

We verify that $\alpha\mathbf{x} \notin W$:

- Suppose it were true that $\alpha\mathbf{x} \in W$.

  We have $\alpha\mathbf{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Since $\alpha\mathbf{x} \in W$, we would have $-1 \geq 0$.

  Contradiction arises.

(d) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \text{There exist some } s, t \in \mathbb{R} \text{ such that } s \geq 0,\ t \geq 0 \text{ and } \mathbf{x} = \begin{bmatrix} s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. [Reminder: If W were a subspace or $\mathbb{R}^2$, then whenever $\mathbf{x} \in W$ and $\alpha \in \mathbb{R}$, it would happen that $\alpha\mathbf{x} \in W$.]

Take $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We have $1 \geq 0$. Then $\mathbf{x} \in W$.

Take $\alpha = -1$. We have $\alpha \in \mathbb{R}$.

[Reminder: If W were a subspace of $\mathbb{R}^2$, then $\alpha\mathbf{x} \in W$.]

We verify that $\alpha\mathbf{x} \notin W$:

- Suppose it were true that $\alpha\mathbf{x} \in W$.

  We have $\alpha\mathbf{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Since $\alpha\mathbf{x} \in W$, we would have $-1 \geq 0$.

  Contradiction arises.

(e) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \textit{There exist some } s, t \in \mathbb{R} \textit{ such that } st \geq 0 \textit{ and } \mathbf{x} = \begin{bmatrix} s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. [Reminder: If W were a subspace or $\mathbb{R}^2$, then whenever $\mathbf{x}, \mathbf{y} \in W$, it would happen that $\mathbf{x} + \mathbf{y} \in W$.]

Take $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

We have $1 \cdot 0 = 0 \geq 0$. Then $\mathbf{x} \in W$. We have $0 \cdot (-1) = 0 \geq 0$. Then $\mathbf{y} \in W$.

[Reminder: If W were a subspace of $\mathbb{R}^2$, then $\mathbf{x} + \mathbf{y} \in W$.]

We verify that $\mathbf{x} + \mathbf{y} \notin W$:

∗ Suppose it were true that $\mathbf{x} + \mathbf{y} \in W$.

  We have $\mathbf{x} + \mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Since $\mathbf{x} + \mathbf{y} \in W$, we would have $1 \cdot (-1) \geq 0$. Then $-1 \geq 0$.

  Contradiction arises.

(f) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{0}, \textit{ or there exist some } s, t \in \mathbb{R} \textit{ such that } s \neq t \textit{ and } \mathbf{x} = \begin{bmatrix} s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. [Reminder: If W were a subspace or $\mathbb{R}^2$, then whenever $\mathbf{x}, \mathbf{y} \in W$, it would happen that $\mathbf{x} + \mathbf{y} \in W$.]

Take $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$.

We have $1 \neq 0$. Then $\mathbf{x} \in W$ and $\mathbf{y} \in W$.

[Reminder: If W were a subspace of $\mathbb{R}^2$, then $\mathbf{x} + \mathbf{y} \in W$.]

We verify that $\mathbf{x} + \mathbf{y} \notin W$:

∗ Suppose it were true that $\mathbf{x} + \mathbf{y} \in W$.

  We have $\mathbf{x} + \mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \mathbf{0}$.

  Since $\mathbf{x} + \mathbf{y} \in W$, we would have $1 \neq 1$.
  Contradiction arises.

(g) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \textit{There exist some } s, t \in \mathbb{R} \textit{ such that } s^2 + t^2 = 1 \textit{ and } \mathbf{x} = \begin{bmatrix} s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. We have $\mathbf{0} \notin W$. (Fill in the detail.)

(h) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \textit{There exist some } s, t \in \mathbb{R} \textit{ such that } s^2 - t^2 = 1 \textit{ and } \mathbf{x} = \begin{bmatrix} s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. We have $\mathbf{0} \notin W$. (Fill in the detail.)

(i) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \textit{There exist some } s, t \in \mathbb{R} \textit{ such that } s = t^2 \textit{ and } \mathbf{x} = \begin{bmatrix} s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. Take $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We have $\mathbf{x} \in W$ and $2\mathbf{x} \notin W$. (Fill in the detail.)

(j) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^2 : \textit{There exist some } s, t \in \mathbb{R} \textit{ such that } s^2 = t^2 \textit{ and } \mathbf{x} = \begin{bmatrix} s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^2$.

Justification. Take $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We have $\mathbf{x}, \mathbf{y} \in W$ and $\mathbf{x} + \mathbf{y} \notin W$. (Fill in the detail.)

5. **Non-examples of subspaces of $\mathbb{R}^3$, from sets of vectors of $\mathbb{R}^3$.**

(a) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^3 : \textit{There exists some } r,s,t \in \mathbb{R} \textit{ such that } r^2+s^2+t^2=1 \textit{ and } \mathbf{x} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^3$.

Justification. We have $\mathbf{0} \notin W$. (Fill in the detail.)

(b) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^3 : \textit{There exists some } r,s,t \in \mathbb{R} \textit{ such that } r = st \textit{ and } \mathbf{x} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^3$.

Justification. Take $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\alpha = 2$.

We have $\mathbf{x} \in W$ and $\alpha\mathbf{x} \notin W$. (Fill in the detail.)

Remark. How do we arrive at this choice of $\mathbf{x}$ and α? This is actually the result of some exploration of the likes below:

- *Suppose W were a subspace of $\mathbb{R}^3$.*

  An arbitrary vector in W would be given by $\begin{bmatrix} r \\ s \\ t \end{bmatrix}$, in which r,s,t were some numbers satisfying $r = st$.

  Then we would expect from (S3) that for each number α, the vector $\begin{bmatrix} \alpha r \\ \alpha s \\ \alpha t \end{bmatrix}$ was a vector in W as well.

  Because it was a vector in W, we would expect the equality $\alpha r = \alpha s \cdot \alpha t$ to hold as well.
  Now ask:
  - ∗ *'In general', when '$r = st$' is true, is it guaranteed that for each α, '$\alpha r = \alpha s \cdot \alpha t$' is also true?*
  - ∗ *Can we name some real numbers r,s,t,α for which $r = st$ and $\alpha r \neq \alpha s \cdot \alpha t$?*

(c) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^3 : \textit{There exists some } r,s,t \in \mathbb{R} \textit{ such that } r^2 = st \textit{ and } \mathbf{x} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^3$.

Justification. Take $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

We have $\mathbf{x} \in W$ and $\mathbf{y} \in W$ and $\mathbf{x} + \mathbf{y} \notin W$. (Fill in the detail.)

Remark. How do we arrive at this choice of $\mathbf{x}$ and $\mathbf{y}$? This is actually the result of some exploration of the likes below:

- *Suppose W were a subspace of $\mathbb{R}^3$.*

  An arbitrary pair of vectors in W would be given by $\begin{bmatrix} r \\ s \\ t \end{bmatrix}$, $\begin{bmatrix} f \\ g \\ h \end{bmatrix}$, in which r,s,t,f,g,h were some numbers satisfying $r^2 = st$ and $f^2 = gh$.

  Then we would expect from (S2) that $\begin{bmatrix} r+f \\ s+g \\ t+h \end{bmatrix}$ was a vector in W as well. Because it was a vector in W, we would expect the equality $(r+f)^2 = (s+g)\cdot(t+h)$ to hold as well.
  Now ask:
  - ∗ *'In general', when '$r^2 = st$' and '$f^2 = gh$' are true, is it guaranteed that '$(r+f)^2 = (s+g)(t+h)$' is also true?*
  - ∗ *Can we name some real numbers r,s,t,f,g,h for which $r^2 = st$ and $f^2 = gh$ and $(r+f)^2 \neq (s+g)(t+h)$?*

(d) *Let* $W = \left\{ \mathbf{x} \in \mathbb{R}^3 : \textit{There exists some } r,s,t \in \mathbb{R} \textit{ such that } r^2 = s^2+t^2 \textit{ and } \mathbf{x} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} \right\}$.

Claim: W is not a subspace of $\mathbb{R}^3$.

Justification. Take $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

We have $\mathbf{x} \in W$ and $\mathbf{y} \in W$ and $\mathbf{x} + \mathbf{y} \notin W$. (Fill in the detail.)

Remark. How do we arrive at this choice of $\mathbf{x}$ and $\mathbf{y}$? This is actually the result of some exploration of the likes below:

- *Suppose W were a subspace of $\mathbb{R}^3$.*

  An arbitrary pair of vectors in W would be given by $\begin{bmatrix} r \\ s \\ t \end{bmatrix}, \begin{bmatrix} f \\ g \\ h \end{bmatrix}$, in which r, s, t, f, g, h were some numbers satisfying $r^2 = s^2 + t^2$ and $f^2 = g^2 + h^2$.

  Then we would expect from (S2) that $\begin{bmatrix} r+f \\ s+g \\ t+h \end{bmatrix}$ was a vector in W as well. Because it was a vector in W, we would expect the equality $(r+f)^2 = (s+g)^2 + (t+h)^2$ to hold as well.

  Now ask:

  * *'In general', when '$r^2 = s^2 + t^2$' and '$f^2 = g^2 + h^2$' are true, is it guaranteed that '$(r+f)^2 = (s+g)^2 + (t+h)^2$' is also true?*
  * *Can we name some real numbers r, s, t, f, g, h for which $r^2 = s^2 + t^2$ and $f^2 = g^2 + h^2$ and $(r+f)^2 \neq (s+g)^2 + (t+h)^2$?*

6. **Non-examples of subspaces of $\mathbb{R}^4$, from sets of vectors of $\mathbb{R}^4$.**

   Give the justification for the claims below.

   In practice, it is easiest to first start with seeing whether (S1) fails to hold. If necessary, continue with seeing whether (S3) fails to hold. If necessary, continue with seeing whether (S2) fails to hold.

   (a) Let $W = \left\{ \mathbf{x} \in \mathbb{R}^4 : \text{There exists some } a, b, c, d \in \mathbb{R} \text{ such that } a + b + c + d = 1 \text{ and } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$.

   W is not a subspace of $\mathbb{R}^4$.

   (b) Let $W = \left\{ \mathbf{x} \in \mathbb{R}^4 : \text{There exists some } a, b, c, d \in \mathbb{R} \text{ such that } a^2 - b^2 + c^2 - d^2 = 1 \text{ and } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$.

   W is not a subspace of $\mathbb{R}^4$.

   (c) Let $W = \left\{ \mathbf{x} \in \mathbb{R}^4 : \text{There exists some } a, b, c, d \in \mathbb{R} \text{ such that } a = bcd \text{ and } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$.

   W is not a subspace of $\mathbb{R}^4$.

   (d) Let $W = \left\{ \mathbf{x} \in \mathbb{R}^4 : \text{There exists some } a, b, c, d \in \mathbb{R} \text{ such that } a^2 = bcd \text{ and } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$.

   W is not a subspace of $\mathbb{R}^4$.

   (e) Let $W = \left\{ \mathbf{x} \in \mathbb{R}^4 : \text{There exists some } a, b, c, d \in \mathbb{R} \text{ such that } a + b = c^2 + d^2 \text{ and } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$.

   W is not a subspace of $\mathbb{R}^4$.

   (f) Let $W = \left\{ \mathbf{x} \in \mathbb{R}^4 : \text{There exists some } a, b, c, d \in \mathbb{R} \text{ such that } ab = cd \text{ and } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$.

   W is not a subspace of $\mathbb{R}^4$.

   (g) Let $W = \left\{ \mathbf{x} \in \mathbb{R}^4 : \text{There exists some } a, b, c, d \in \mathbb{R} \text{ such that } a^2 + b^2 = c^2 + d^2 \text{ and } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$.

   W is not a subspace of $\mathbb{R}^4$.

   (h) Let $W = \left\{ \mathbf{x} \in \mathbb{R}^4 : \text{There exists some } a, b, c, d \in \mathbb{R} \text{ such that } abcd = 0 \text{ and } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$.

   W is not a subspace of $\mathbb{R}^4$.