1. **Definition. (Column space of a matrix.)**

Let H be a $(p \times q)$-matrix.

The column space of the matrix H is defined to be the set

$$\left\{ \mathbf{y} \in \mathbb{R}^p : \begin{array}{l} \textit{There exist some } \mathbf{u} \in \mathbb{R}^q \\ \textit{such that } \mathbf{y} = H\mathbf{u}. \end{array} \right\}$$

We denote this set by $\mathcal{C}(H)$.

Remark. We are applying the method of specification, with 'selection criterion'

(∗) '*there exist some $\mathbf{u} \in \mathbb{R}^q$ such that $\mathbf{y} = H\mathbf{u}$.*'

to form a certain set of vectors in $\mathbb{R}^p$, called the column space of the matrix H.

When put into plain words, the selection criterion (∗) reads:

> '*y is a vector in $\mathbb{R}^p$ which can be expressed as the product of H in the left and some vector in $\mathbb{R}^q$ in the right.*'

According to this ‘selection criterion’

- Those vectors in $\mathbb{R}^p$ resultant from multiplying H from the left to some vector in $\mathbb{R}^q$ are collected.
- Those vectors in $\mathbb{R}^p$ not resultant from multiplying H from the left to some vector in $\mathbb{R}^q$ are ‘discarded’.

For this reason, $\mathcal{C}(H)$ is simply the collection of all vectors in $\mathbb{R}^p$ which can be ‘expressed in the form’ $H\mathbf{u}$, and only such vectors.

So very often the set $\mathcal{C}(H)$ is given the short-hand $\{H\mathbf{u} \mid \mathbf{u} \in \mathbb{R}^q\}$.

Further remark.

How to use the various versions of the definitions?

Always remember, whenever $\mathbf{v} \in \mathbb{R}^p$, the statements below mean the same thing:

(a) $\mathbf{v} \in \mathcal{C}(H)$.

(b) *There exists some* $\mathbf{u} \in \mathbb{R}^q$ *such that* $\mathbf{v} = H\mathbf{u}$.

2. **Theorem (1). (Column space of a matrix as a 'subspace'.)**

Suppose H is a $(p \times q)$-matrix. Then the statements below hold.

(a) $\mathbf{0}_p \in \mathcal{C}(H)$.

(b) *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$, *if* $\mathbf{x} \in \mathcal{C}(H)$ *and* $\mathbf{y} \in \mathcal{C}(H)$ *then* $\mathbf{x} + \mathbf{y} \in \mathcal{C}(H)$.

(c) *For any* $\mathbf{x} \in \mathbb{R}^p$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{x} \in \mathcal{C}(H)$ *then* $\alpha\mathbf{x} \in \mathcal{C}(H)$.

(d) *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$, *for any* $\alpha, \beta \in \mathbb{R}$, *if* $\mathbf{x} \in \mathcal{C}(H)$ *and* $\mathbf{y} \in \mathcal{C}(H)$ *then* $\alpha\mathbf{x} + \beta\mathbf{y} \in \mathcal{C}(H)$.

3. **Proof of Theorem (1).**

Suppose H is a $(p \times q)$-matrix.

(a) Note that $\mathbf{0}_p = H\mathbf{0}_q$, and $\mathbf{0}_q \in \mathbb{R}^q$.

Then $\mathbf{0}_p \in \mathcal{C}(H)$.

(b) Pick any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$. Suppose $\mathbf{x}, \mathbf{y} \in \mathcal{C}(H)$.

[Ask: What to verify? Answer: 'There exist some $\mathbf{w} \in \mathbb{R}^q$ such that $\mathbf{x} + \mathbf{y} = H\mathbf{w}$.'
Further ask: How comes such a vector $\mathbf{w}$? Answer: Make use of the information provided by '$\mathbf{x} \in \mathcal{C}(H)$' and '$\mathbf{y} \in \mathcal{C}(H)$'.]

By definition of $\mathcal{C}(H)$, there exist some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^q$ such that $\mathbf{x} = H\mathbf{u}$ and $\mathbf{y} = H\mathbf{v}$.

Now $\mathbf{x} + \mathbf{y} = H\mathbf{u} + H\mathbf{v} = H(\mathbf{u} + \mathbf{v})$. Since $\mathbf{u}, \mathbf{v} \in \mathbb{R}^q$, it happens that $\mathbf{u} + \mathbf{v} \in \mathbb{R}^q$.

Then by the definition of $\mathcal{C}(H)$, $\mathbf{x} + \mathbf{y} \in \mathcal{C}(H)$.

(c) Pick any $\mathbf{x} \in \mathbb{R}^p$. Pick any $\alpha \in \mathbb{R}$. Suppose $\mathbf{x} \in \mathcal{C}(H)$.

[Ask: What to verify?
Answer. 'There exist some $\mathbf{w} \in \mathbb{R}^q$ such that $\alpha\mathbf{x} = H\mathbf{w}$ ']

By definition of $\mathcal{C}(H)$, there exist some $\mathbf{u} \in \mathbb{R}^q$ such that $\mathbf{x} = H\mathbf{u}$.

Now $\alpha\mathbf{x} = \alpha H\mathbf{u} = H(\alpha\mathbf{u})$. Since $\mathbf{u}, \mathbf{v} \in \mathbb{R}^q$, it happens that $\alpha\mathbf{u} \in \mathbb{R}^q$.

Then by the definition of $\mathcal{C}(H)$, $\alpha\mathbf{x} \in \mathcal{C}(H)$.

(d) Exercise.

4. An alternative way of visualizing the notion of *column space* is through the notions of *linear combination* and *span* (which will be introduced shortly).

Recall the definition for the notion of *linear combination*:

Let $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *be vectors in* $\mathbb{R}^m$*.*

Let $\mathbf{w}$ *be a vector in* $\mathbb{R}^m$*.*

We say $\mathbf{v}$ *is a linear combination of* $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *if the statement* (†) *holds:*

(†) There exist some real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $\mathbf{w} = \alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 + \cdots + \alpha_n\mathbf{z}_n$.

The expression $\alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 + \cdots + \alpha_n\mathbf{z}_n$ on its own is called the linear combination of the vectors $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ and the scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

5. **Definition. (Span of a set of vectors in $\mathbb{R}^m$.)**

Let $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *be ('finitely many') vectors in* $\mathbb{R}^m$.

The span of (the set of vectors) $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *is defined to be the set*

$$\left\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} \textit{ is a linear combination of } \mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n \right\}$$

We denote this set by $\mathsf{Span}\,(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$ *(or* $\langle\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\}\rangle$*).*

Remark.

$\mathsf{Span}\,(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$ is constructed with the help of the method of specification, with 'selection criterion'

($\star$) '$\mathbf{y}$ *is a linear combination of* $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$,'

when we collect those and only those vectors in $\mathbb{R}^m$ which are linear combinations of $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$.

For this reason, $\mathsf{Span}\,(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$ is simply the collection of all vectors in $\mathbb{R}^m$ which can be 'expressed' as linear combinations of $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$, and only such vectors.

Further remark.

How to use the various versions of the definitions?

Always remember, whenever $\mathbf{v} \in \mathbb{R}^m$, the statements below mean the same thing:

($\sharp$) $\mathbf{y}$ *belongs to* $\mathsf{Span}\ (\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$.

($\natural$) $\mathbf{y}$ *is a linear combination of* $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$.

($\flat$) *There exist some real numbers* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *such that* $\mathbf{y} = \alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 + \cdots + \alpha_n\mathbf{z}_n$.

Further remark on terminologies and symbols.

(a) In some textbooks, it is emphasized that the notion of *span* is defined on sets of vectors; hence the brackets '$\{$', '$\}$' are used in the notation.

(b) For convenience, we may read '$\mathbf{y} \in \mathsf{Span}\ (\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$' as '$\mathbf{y}$ *is spanned by* $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$'. When a set of vectors, say, V, is equal to the set $\mathsf{Span}\ (\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$, we may read this set equality as '*the set V is spanned by* $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$'.

6. With the help of Lemma (A) (from the handout *linear combinations*), we are going to set up a 'dictionary' between the notion of *span* and the notion of *column space.*

Recall Lemma (A):

Let A be an $(m \times n)$-matrix, and $\mathbf{t}$ be a vector in $\mathbb{R}^n$.

Suppose that for each $j = 1, 2, \cdots, n$, the j-th column of A is $\mathbf{a}_j$ and the j-th entry of $\mathbf{t}$ is t_j. (So $A = \left[\, \mathbf{a}_1 \,\middle|\, \mathbf{a}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{a}_n \,\right]$ and $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$.)

Then $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$.

7. **Theorem (D). ('Dictionary' between the notion of span and the notion of column space.)**

Let $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q$ *be vectors in* $\mathbb{R}^p$, *and* H *be a* $(p \times q)$*-matrix.*

Suppose that the j*-th column of* H *is* $\mathbf{h}_j$ *for each* j. *(So* $H = \left[\, \mathbf{h}_1 \,\middle|\, \mathbf{h}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{h}_q \,\right]$*.)*

Then $\mathcal{C}(H) = \mathsf{Span}\,(\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\})$.

Remark.

The significance of Theorem (D) is that every statement about spans of collections of finitely many vectors can be translated into a statement about column spaces of matrices, and vice versa.

Further remark.

The equality '$\mathcal{C}(H) = \mathsf{Span}\,(\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\})$' is a set equality. What such an equality means is that the statements (†), (‡) below hold simultaneously:

(†) For any $\mathbf{y} \in \mathbb{R}^p$, if $\mathbf{y} \in \mathcal{C}(H)$ then $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\})$.

(‡) For any $\mathbf{y} \in \mathbb{R}^p$, if $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\})$ then $\mathbf{y} \in \mathcal{C}(H)$.

8. **Proof of Theorem (D).**

Let $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q$ be vectors in $\mathbb{R}^p$, and H be a $(p \times q)$-matrix.

Suppose that the j-th column of H is $\mathbf{h}_j$ for each j. Then $H = \left[\ \mathbf{h}_1 \middle| \mathbf{h}_2 \middle| \cdots \middle| \mathbf{h}_q\ \right]$.

[We verify the statements (†), (‡):

(†) For any $\mathbf{y} \in \mathbb{R}^p$, if $\mathbf{y} \in \mathcal{C}(H)$ then $\mathbf{y} \in \mathsf{Span}\ (\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\})$.

(‡) For any $\mathbf{y} \in \mathbb{R}^p$, if $\mathbf{y} \in \mathsf{Span}\ (\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\})$ then $\mathbf{y} \in \mathcal{C}(H)$.

The arguments are given in two separate paragraphs, one for (†) and the other (‡).]

- Pick any $\mathbf{y} \in \mathbb{R}^p$. Suppose $\mathbf{y} \in \mathcal{C}(H)$.
  Then there exists some $\mathbf{u} \in \mathbb{R}^q$ such that $\mathbf{y} = H\mathbf{u}$.
  For each i, denote the i-th entry of $\mathbf{u}$ by u_i.
  Then, by Lemma (A), $\mathbf{y} = u_1\mathbf{h}_1 + u_2\mathbf{h}_2 + \cdots + u_q\mathbf{h}_q$.
  Therefore $\mathbf{y} \in \mathsf{Span}\ (\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\})$

- Pick any $\mathbf{y} \in \mathbb{R}^p$. Suppose $\mathbf{y} \in \mathsf{Span}\ (\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\})$.

  Then there exists some $u_1, u_2, \cdots, u_q \in \mathbb{R}$ such that $\mathbf{y} = u_1\mathbf{h}_1 + u_2\mathbf{h}_2 + \cdots + u_q\mathbf{h}_q$.

  Define the vector $\mathbf{u}$ in $\mathbb{R}^q$ by $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix}$.

  Then by Lemma (A), we have $\mathbf{y} = H\mathbf{u}$.

  Therefore $\mathbf{y} \in \mathcal{C}(H)$.

It follows that $\mathcal{C}(H) = \mathsf{Span}\ (\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_q\})$ holds.

9. **Illustrations of the content of Theorem (D).**

(a) $$\mathcal{C}\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix}\right) = \mathsf{Span}\left(\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}\right\}\right)$$

(b) $$\mathcal{C}\left(\begin{bmatrix} 1 & 0 & 9 \\ 0 & 2 & 8 \\ 1 & 4 & 7 \\ 0 & 6 & 6 \\ 1 & 8 & 5 \end{bmatrix}\right) = \mathsf{Span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 7 \\ 6 \\ 5 \end{bmatrix}\right\}\right)$$

10. **Theorem (2). (Span of vectors as a 'subspace'.)**

Suppose $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *are vectors in* $\mathbb{R}^m$. *Write* $V = \mathsf{Span}\left(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\}\right)$.

The statements below hold:

(a) $\mathbf{0} \in V$.

(b) *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, *if* $\mathbf{x} \in V$ *and* $\mathbf{y} \in V$ *then* $\mathbf{x} + \mathbf{y} \in V$.

(c) *For any* $\mathbf{x} \in \mathbb{R}^m$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{x} \in V$ *then* $\alpha\mathbf{x} \in V$.

(d) *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, *for any* $\alpha, \beta \in \mathbb{R}$, *if* $\mathbf{x} \in V$ *and* $\mathbf{y} \in V$ *then* $\alpha\mathbf{x} + \beta\mathbf{y} \in V$.

Proof of Theorem (2). This is a consequence of Theorem (1) and Theorem (D).