

1. According to Lemma ( $\gamma$ ) and Theorem (C):

- (a) Each row-operation matrix is non-singular and invertible.
- (b) The product of any (finitely many) row-operation matrices is non-singular and invertible.
- (c) Every non-singular (and invertible) matrix is a product of row-operation matrices.

This allows us to establish a ‘dictionary’ between row-equivalence for general matrices and non-singularity.

2. **Lemma ( $\mu$ ).**

Let  $A, B$  be  $(m \times n)$ -matrices. Suppose  $A$  is row-equivalent to  $B$ . Then there exists some non-singular and invertible  $(m \times m)$ -square matrix  $H$  such that  $B = HA$ .

**Proof of Lemma ( $\mu$ ).**

Let  $A, B$  be  $(m \times n)$ -matrices. Suppose  $A$  is row-equivalent to  $B$ .

Then there is a sequence of row operations joining  $A$  to  $B$ :

$$A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} C_3 \xrightarrow{\rho_3} \cdots \xrightarrow{\rho_{N-1}} C_N = B$$

in which the  $\rho_j$ 's are the various row operations involved in this sequence.

For each  $k$ , denote by  $H_k$  the row operation matrix corresponding to  $\rho_k$ .

Then  $C_2 = H_1 C_1 = H_1 A$ ,  $C_3 = H_2 C_2$ , ...,  $C_{N-1} = H_{N-2} C_{N-2}$ , and  $B = C_N = H_{N-1} C_{N-1}$ .

Therefore  $B = HA$ , in which  $H = H_{N-1} H_{N-2} \cdots H_2 H_1$ .

Then by Lemma ( $\gamma$ ) and Theorem (C), the matrix  $H$  is non-singular and invertible.

3. **Lemma ( $\nu$ ).**

Let  $A, B$  be  $(m \times n)$ -matrices. Suppose there exists some non-singular and invertible  $(m \times m)$ -square matrix  $H$  such that  $B = HA$ .

Then  $A$  is row-equivalent to  $B$ .

**Proof of Lemma ( $\nu$ ).**

Let  $A, B$  be  $(m \times n)$ -matrices. Suppose there exists some non-singular and invertible  $(m \times m)$ -square matrix  $H$  such that  $B = HA$ .

According to Theorem (C), there exist some exist some  $(m \times m)$ -row-operation matrices  $H_1, H_2, \dots, H_{N-1}$  such that  $H = H_{N-1} H_{N-2} \cdots H_2 H_1$ .

Define  $C_1 = A$ ,  $C_2 = H_1 C_1$ ,  $C_3 = H_2 C_2$ , ...,  $C_{N-1} = H_{N-2} C_{N-2}$ ,  $C_N = H_{N-1} C_{N-1}$ .

Then, by definition,  $B = HA = H_{N-1} H_{N-2} \cdots H_2 H_1 C_1 = H_{N-1} H_{N-2} \cdots H_2 C_2 = \cdots = H_{N-1} C_{N-1} = C_N$ .

Denote the row operations corresponding to the respective row-operation matrices  $H_1, H_2, \dots, H_{N-1}$  by  $\rho_1, \rho_2, \dots, \rho_{N-1}$ .

Then, by definition,  $A, C_2, C_3, \dots, C_{N-1}, B$  are joint by the row operations

$$A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} C_3 \xrightarrow{\rho_3} \cdots \xrightarrow{\rho_{N-1}} C_N = B.$$

It follows that  $A$  is row-equivalent to  $B$ .

4. We combine Lemma ( $\mu$ ) and Lemma ( $\nu$ ) into Theorem (F) below.

**Theorem (F). (Re-formulation of row-equivalence in terms of multiplication by non-singular and invertible matrices.)**

Let  $A, B$  be  $(m \times n)$ -matrices.

The statements below are logically equivalent:

- (a)  $A$  is row-equivalent to  $B$ .
- (b) There exists some non-singular and invertible  $(m \times m)$ -square matrix  $H$  such that  $B = HA$ .

**Remark.** Such a re-formulation of row equivalence is useful in theoretical discussions because it brings in the equality symbol ‘=’.

5. **Illustrations.**

(a) Let  $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ .

It happens that

$$A = C_1 \xrightarrow{-1R_1+R_2} C_2 \xrightarrow{-2R_1+R_3} C_3 \xrightarrow{-2R_2+R_3} C_4 \xrightarrow{-1R_3} C_5 \xrightarrow{-2R_2+R_1} C_6 \xrightarrow{-1R_3+R_2} C_7 = B$$

Then  $B = HA$ , in which  $H$  is the  $(3 \times 3)$ -square matrix given by the product  $H = H_6H_5H_4H_3H_2H_1$ , and for each  $k = 1, 2, 3, 4, 5, 6$ , the matrix  $H_k$  is the row operation matrix corresponding to the row operation  $\rho_k$  joining  $C_k$  to  $C_{k+1}$ .

$H$  is obtained as a resultant of the application of the sequence of row operations  $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$  on  $I_3$ :

$$\begin{aligned} I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &\xrightarrow{-1R_1+R_2} H_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{-2R_1+R_3} H_2H_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{-2R_2+R_3} H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \\ &\xrightarrow{-1R_3} H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \\ &\xrightarrow{-2R_2+R_1} H_5H_4H_3H_2H_1 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \\ &\xrightarrow{-1R_3+R_2} H = H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \end{aligned}$$

(b) Let  $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & -2 & 3 & -4 \\ 2 & 7 & -12 & 11 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ .

It happens that

$$A = C_1 \xrightarrow{R_1 \leftrightarrow R_2} C_2 \xrightarrow{-1R_1} C_3 \xrightarrow{-2R_1+R_3} C_4 \xrightarrow{-3R_2+R_3} C_5 \xrightarrow{-2R_2+R_1} C_6 = B$$

Then  $B = HA$ , in which  $H$  is the  $(3 \times 3)$ -square matrix given by the product  $H = H_5H_4H_3H_2H_1$ , and for each  $k = 1, 2, 3, 4, 5$ , the matrix  $H_k$  is the row operation matrix corresponding to the row operation  $\rho_k$  joining  $C_k$  to  $C_{k+1}$ .

$H$  is obtained as a resultant of the application of the sequence of row operations  $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$  on  $I_3$ :

$$\begin{aligned} I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{-1R_1} H_2H_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{-2R_1+R_3} H_3H_2H_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\ &\xrightarrow{-3R_2+R_3} H_4H_3H_2H_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \\ &\xrightarrow{-2R_2+R_1} H = H_5H_4H_3H_2H_1 = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \end{aligned}$$

(c) Let  $A = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$ .

It happens that

$$A = C_1 \xrightarrow{R_1 \leftrightarrow R_2} C_2 \xrightarrow{2R_1+R_3} C_3 \xrightarrow{-3R_2+R_3} C_4 \xrightarrow{-2R_2+R_1} C_5 \xrightarrow{2R_3+R_1} C_6 \xrightarrow{-2R_3+R_2} C_7 = B$$

Then  $B = HA$ , in which  $H$  is the  $(3 \times 3)$ -square matrix given by the product  $H = H_6H_5H_4H_3H_2H_1$ , and for each  $k = 1, 2, 3, 4, 5, 6$ , the matrix  $H_k$  is the row operation matrix corresponding to the row operation  $\rho_k$  joining  $C_k$  to  $C_{k+1}$ .

$H$  is obtained as a resultant of the application of the sequence of row operations  $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$  on  $I_3$ :

$$\begin{aligned}
I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\xrightarrow{2R_1 + R_3} H_2H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\
&\xrightarrow{-3R_2 + R_3} H_3H_2H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \\
&\xrightarrow{-2R_2 + R_1} H_4H_3H_2H_1 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \\
&\xrightarrow{2R_3 + R_1} H_5H_4H_3H_2H_1 = \begin{bmatrix} -8 & 5 & 2 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \\
&\xrightarrow{-2R_3 + R_2} H = H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} -8 & 5 & 2 \\ 7 & -4 & -2 \\ -3 & 2 & 1 \end{bmatrix}
\end{aligned}$$

(d) Let  $A = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ .

It happens that

$$\begin{aligned}
A = C_1 &\xrightarrow{-1R_1 + R_2} C_2 \xrightarrow{-3R_1 + R_3} C_3 \xrightarrow{-1R_1 + R_4} C_4 \xrightarrow{-1R_2} C_5 \\
&\xrightarrow{4R_2 + R_3} C_6 \xrightarrow{3R_2 + R_4} C_7 \xrightarrow{R_3 \leftrightarrow R_4} C_8 \xrightarrow{4R_3 + R_4} C_8 \xrightarrow{-2R_2 + R_1} C_9 \xrightarrow{-1R_3 + R_1} C_{10} = B
\end{aligned}$$

Then  $B = HA$ , in which  $H$  is the  $(4 \times 4)$ -square matrix given by the product  $H = H_{10}H_9 \cdots H_3H_2H_1$ , and for each  $k = 1, 2, 3, \dots, 9, 10$ , the matrix  $H_k$  is the row operation matrix corresponding to the row operation  $\rho_k$  joining  $C_k$  to  $C_{k+1}$ .

$H$  is obtained as a resultant of the application of the sequence of row operations  $\rho_1, \rho_2, \rho_3, \dots, \rho_9, \rho_{10}$  on  $I_4$ :

$$\begin{aligned}
I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} &\xrightarrow{-1R_1+R_2} H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\xrightarrow{-3R_1+R_3} H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\xrightarrow{-1R_1+R_4} H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \\
&\xrightarrow{-1R_2} H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \\
&\xrightarrow{4R_2+R_3} H_5H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \\
&\xrightarrow{3R_2+R_4} H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix} \\
&\xrightarrow{R_3 \leftrightarrow R_4} H_7H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -4 & 1 & 0 \end{bmatrix} \\
&\xrightarrow{4R_3+R_4} H_8H_7H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 9 & -16 & 1 & 4 \end{bmatrix} \\
&\xrightarrow{-2R_2+R_1} H_9H_8H_7H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 9 & -16 & 1 & 4 \end{bmatrix} \\
&\xrightarrow{-1R_3+R_1} H = H_{10}H_9H_8H_7H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} -3 & 5 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 9 & -16 & 1 & 4 \end{bmatrix}
\end{aligned}$$