MATH1030 Non-singularity and invertibility

1. **Definition. (Invertibility.)**

   Let A be an $(n \times n)$-square matrix.

   (a) *Suppose B is a $(n \times n)$-square matrix. Further suppose $BA = I_n$ and $AB = I_n$. Then we say B is a matrix inverse of A.*
   (b) *A is said to be invertible if and only if A has a matrix inverse.*

2. **Two (trivial) examples.**

   (a) The identity matrix I_n is invertible, and a matrix inverse of it is I_n itself.
   (b) The zero $(n \times n)$-square matrix is not invertible.

3. **Lemma (α). (Uniqueness of matrix inverse.)**

   Let A be an $(n \times n)$-square matrix. Suppose B, C are both matrix inverses of A. Then $B = C$.

   Proof of Lemma (α).

   Under the assumption, we have $BA = I_n$ and $AC = I_n$. Then $B = BI_n = B(AC) = (BA)C = I_nC = C$.

   Remarks.

   - From now on there is no problem using the article *the* in writing the words *the matrix inverse of the invertible matrix blah-blah-blah.*
   - For the same reason, it makes to label the matrix inverse of an invertible matrix, say, A, with something which involves the symbol 'A'.

     From now on, we denote by A^{-1} the matrix inverse of such an invertible matrix A.

4. **Lemma (β). (Product of matrix inverses.)**

   Let A, B be $(n \times n)$-square matrices.

   Suppose A, B are invertible. Then the product AB is invertible with matrix inverse given by $(AB)^{-1} = B^{-1}A^{-1}$.

   Proof of Lemma (β).

   Under the assumption, we have $A^{-1}A = I_n$ and $AA^{-1} = I_n$. Moreover, $B^{-1}B = I_n$ and $BB^{-1} = I_n$.

   Write $C = AB$, and $D = B^{-1}A^{-1}$.

   We have $DC = (B^{-1}A^{-1})(AB) = B^{-1}[A^{-1}(AB)] = B^{-1}[(A^{-1}A)B] = B^{-1}(I_nB) = B^{-1}B = I_n$.

   We also have $CD = (AB)(B^{-1}A^{-1}) = \cdots\cdots\cdots = I_n$.

   Therefore, by the definition of matrix inverse and invertibility, C is invertible with matrix inverse D.

   Then AB is invertible, and its matrix inverse is given by $(AB)^{-1} = B^{-1}A^{-1}$.

   Remark. By mathematical induction, we can prove this generalization of Lemma (β):

   > *Let $A_1, A_2, \cdots, A_k$ be $(n \times n)$-square matrices.*
   > *Suppose $A_1, A_2, \cdots, A_k$ are invertible. Then the product $A_1A_2\cdots A_k$ is invertible with matrix inverse given by $(A_1A_2\cdots A_k)^{-1} = {A_k}^{-1}\cdots{A_2}^{-1}{A_1}^{-1}$.*

5. **Corollary to Lemma (β).**

   Let A be an $(n \times n)$-square matrix. Suppose A is invertible. Then, for each positive integer p, the matrix A^p is invertible with matrix inverse given by $(A^p)^{-1} = (A^{-1})^p$.

   Proof. Exercise in mathematical induction.

6. **Lemma (γ).**

   (a) *Every row operation matrix is invertible. Its matrix inverse is the row operation matrix corresponding to its reverse row operation.*
   (b) *Suppose $H_1, H_2, \cdots, H_k$ are row-operation matrices, and $H = H_k\cdots H_2H_1$.*

   Then H is invertible, and its matrix inverse is given by $H^{-1} = {H_1}^{-1}{H_2}^{-2}\cdots{H_k}^{-1}$.

   Proof of Lemma (γ). [This is a straightforward calculation, though it requires patience.]

   Let C, C' be $(n \times n)$-square matrices. Suppose C' is the resultant of the application of some row operation ρ on C. Denote by $\tilde{\rho}$ the 'reverse' row operation corresponding to ρ. Denote the respective row operation matrices corresponding to ρ, $\tilde{\rho}$ by H, $\tilde{H}$ respectively.

   Then $C' = HC$ and $C = \tilde{H}C'$.

- Suppose ρ is given by $\alpha R_i + R_k$, in which α is some real number, and $i \neq k$. Then $\tilde{\rho}$ is given by $-\alpha R_i + R_k$.
  Therefore $H = I_n + \alpha E^{n,n}_{k,i}$, $\tilde{H} = I_n - \alpha E^{n,n}_{k,i}$.
  Since $i \neq k$, we have $(E^{n,n}_{k,i})^2 = \mathcal{O}_{n\times n}$. Then
  $$\begin{aligned} \tilde{H}H &= (I_n - \alpha E^{n,n}_{k,i})(I_n + \alpha E^{n,n}_{k,i}) \\ &= I_nI_n - (\alpha E^{n,n}_{k,i})I_n + I_n(\alpha E^{n,n}_{k,i}) - (\alpha E^{n,n}_{k,i})(\alpha E^{n,n}_{k,i}) \\ &= I_n - \alpha^2 (E^{n,n}_{k,i})^2 \\ &= I_n - \alpha^2 \mathcal{O}_{n\times n} = I_n \end{aligned}$$
  Similarly, we have $H\tilde{H} = I_n$.
- Suppose ρ is given by βR_k, in which β is some non-zero real number. Then $\tilde{\rho}$ is given by $(1/\beta)R_k$.
  Therefore $H = I_n + (\beta - 1)E^{n,n}_{k,k}$, $\tilde{H} = I_n + (1/\beta - 1)E^{n,n}_{k,k}$.
  We have $(E^{n,n}_{k,k})^2 = E^{n,n}_{k,k}$. Then
  $$\begin{aligned} \tilde{H}H &= [I_n + (\beta - 1)E^{n,n}_{k,k}][I_n + (1/\beta - 1)E^{n,n}_{k,k}] \\ &= I_nI_n + [(\beta - 1)E^{n,n}_{k,k}]I_n + I_n[(1/\beta - 1)E^{n,n}_{k,k}] + [(\beta - 1)E^{n,n}_{k,k}][(1/\beta - 1)E^{n,n}_{k,k}] \\ &= I_n + (\beta + 1/\beta - 2)E^{n,n}_{k,k} + (-\beta - 1/\beta + 2)(E^{n,n}_{k,k})^2 \\ &= I_n + (\beta + 1/\beta - 2)E^{n,n}_{k,k} + (-\beta - 1/\beta + 2)E^{n,n}_{k,k} = I_n \end{aligned}$$
  Similarly, we have $H\tilde{H} = I_n$.
- Suppose ρ is given by $R_i \leftrightarrow R_k$, in which $i \neq k$. Then $\tilde{\rho}$ is given by $R_i \leftrightarrow R_k$.
  Therefore $H = I_n - E^{n,n}_{i,i} - E^{n,n}_{k,k} + E^{n,n}_{i,k} + E^{n,n}_{k,i} = \tilde{H}$.
  Note that:
  * If p, q, r, s are integers between 1 and n, then $E^{n,n}_{p,q}E^{n,n}_{r,s} = \begin{cases} E^{n,n}_{p,s} & \text{when} \quad q = r \\ \mathcal{O}_{n\times n} & \text{when} \quad q \neq r. \end{cases}$

  Since $i \neq k$, we have
  $$\tilde{H}H = H\tilde{H} = H^2 = (I_n - E^{n,n}_{i,i} - E^{n,n}_{k,k} + E^{n,n}_{i,k} + E^{n,n}_{k,i})^2 = \cdots = I_n.$$
  (Fill in the detail.)

The rest of Lemma (γ) follows from the above.

7. **Lemma (δ). (Invertibility of matrix inverse.)**

   Let A be an $(n \times n)$-square matrix.

   Suppose A is invertible. Then its matrix inverse A^{-1} is invertible, and the matrix inverse of A^{-1} is given by $(A^{-1})^{-1} = A$.

   Proof of Lemma (δ).

   Under the assumption, we have $A^{-1}A = I_n$ and $AA^{-1} = I_n$.

   Write $B = A$, and $C = A^{-1}$.

   Since $AA^{-1} = I_n$, we have $BC = I_n$.

   Since $A^{-1}A = I_n$, we have $CB = I_n$. Therefore $BC = I_n$ and $CB = I_n$.

   Then, by the definition of matrix inverse and invertibility, C is invertible with matrix inverse B.

   Therefore A^{-1} is invertible and its matrix inverse is given by $(A^{-1})^{-1} = A$.

8. **Lemma (ϵ). (Invertibility implies non-singularity.)**

   Let A be an $(n \times n)$-square matrix.

   Suppose A invertible. Then A is non-singular, and its matrix inverse A^{-1} is non-singular and invertible.

   Proof of Lemma (ϵ).

   Under the assumption, we have $A^{-1}A = I_n$ and $AA^{-1} = I_n$. By Lemma (δ), A^{-1} is invertible.

   Since $A^{-1}A = I_n$, we conclude from Lemma (2) that A is non-singular.

   Since $AA^{-1} = I_n$, we conclude from Lemma (2) that A^{-1} is non-singular.

   Remark. Recall Lemma (2):

Let C be a $(p \times p)$-square matrix.

Suppose there exists some $(p \times p)$-square matrix J such that $JC = I_p$.

Then C is non-singular.

9. **Lemma (ζ). (Non-singularity implies invertibility.)**

   Let A be an $(n \times n)$-square matrix.

   Suppose A non-singular. Then A is invertible, and its matrix inverse A^{-1} is non-singular and invertible.

   Proof of Lemma (ζ).

   Under the assumption, and according to Lemma (6), there exists some $(n \times n)$-square matrix H such that H is non-singular, $HA = I_n$ and $AH = I_n$.

   Now, by definition, A is invertible with its matrix inverse given by $A^{-1} = H$.

   H is invertible by Lemma (δ).

10. **Theorem (B). (Equivalence of non-singularity and invertibility.)**

    Let A be an $(n \times n)$-square matrix.

    A is non-singular if and only if A is invertible.

    Furthermore, if A is invertible, then its matrix inverse A^{-1} is non-singular and invertible with matrix inverse given by $(A^{-1})^{-1} = A$.

11. **Corollary to Theorem (B).**

    Let A be an $(n \times n)$-square matrix.

    The statements below are logically equivalent:

    (a) *A is invertible.*

    (b) *There exists some $(n \times n)$-square matrix H such that $HA = I_n$.*

    (c) *There exists some $(n \times n)$-square matrix G such that $AG = I_n$.*

    Proof of Corollary to Theorem (B).

    - Suppose A is invertible. Then A has a unique matrix inverse A^{-1}. So it follows that $A^{-1}A = I_n$ and $AA^{-1} = I_n$.
    - Suppose there exists some $(n \times n)$-square matrix H such that $HA = I_n$. Then, by Lemma (2), A is non-singular. Therefore, by Theorem (B), A is invertible.
    - Suppose there exists some $(n \times n)$-square matrix G such that $AG = I_n$. Then, by Lemma (2), G is non-singular. Therefore, by Theorem (B), G is invertible. We verify that $G^{-1} = A$:

      We have $I_n = AG$. Then $G^{-1} = I_n G^{-1} = (AG)G^{-1} = A(GG^{-1}) = AI_n = A$.

      Then by Lemma (δ), A is invertible.

    Remark. With the help of Theorem (B) and its corollary, together with the calculations leading towards Lemma (6), we can 'upgrade' Theorem (A) to obtain Theorem (C).

12. **Theorem (C). (Various re-formulations for the notions of non-singularity and invertibility.)**

    Let A be an $(n \times n)$-square matrix. The statements below are logically equivalent:

    (a) *A is non-singular.*

    (b) *For any vector $\mathbf{v}$ in $\mathbb{R}^n$, if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.*

    (c) *The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.*

    (d) *A is row-equivalent to I_n.*

    (e) *A is invertible.*

    (f) *There exists some $(n \times n)$-square matrix H such that $HA = I_n$.*

    (g) *There exists some $(n \times n)$-square matrix G such that $AG = I_n$.*

Now suppose A is non-singular, with a sequence of row operations

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} C_{p-1} \xrightarrow[\rho_{p-1}]{} C_p = I_n,$$

and with H_k being the row-operation matrix corresponding to ρ_k for each k.

Then $[I_n|A^{-1}]$ is the resultant of the application of the same sequence of row operations $\rho_1, \rho_2, \cdots, \rho_{p-1}$ starting from $[A|I_n]$:

$$[A|I_n] = [C_1|I_n] \xrightarrow[\rho_1]{} [C_2|H_1] \xrightarrow[\rho_2]{} [C_3|H_2H_1] \xrightarrow[\rho_3]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} [C_{p-1}|H_{p-2}\cdots H_2H_1] \xrightarrow[\rho_{p-1}]{} [C_p|H_{p-1}\cdots H_2H_1] = [I_n|A^{-1}].$$

Moreover, A^{-1} and A are respectively given as products of row-operation matrices by

$$A^{-1} = H_{p-1}\cdots H_2H_1, \qquad\qquad A = {H_1}^{-1}{H_2}^{-1}\cdots {H_{p-1}}^{-1}.$$

13. **Corollary to Theorem (C).**

    The statements below hold:

    (a) *The matrix inverse of every invertible matrix is a product of finitely many row-operation matrices.*

    (b) *Every non-singular matrix is a product of finitely many row-operation matrices.*

14. Recall what Lemma (β) says (when put in plain words): the product of any two invertible matrices is an invertible. We now upgrade Lemma (β) with the help of Theorem (B), to obtain Lemma (β').

    Lemma (β').

    Let A, B be $(n \times n)$-square matrices. Suppose A, B are non-singular and invertible. Then the product AB is non-singular and invertible.

15. The converse of Lemma (β'), as formulated below, is also true.

    Lemma (η).

    Let A, B be $(n \times n)$-square matrices. Suppose the product AB is non-singular and invertible. Then each of A, B is non-singular and invertible.

    Proof of Lemma (η).

    Suppose AB is non-singular and invertible.

    We verify that B is non-singular:

    - [This amounts to verifying the statement '*for any* $\mathbf{v} \in \mathbb{R}^n$, *if* $B\mathbf{v} = \mathbf{0}$ *then* $\mathbf{v} = \mathbf{0}$.']
      Pick any $\mathbf{v} \in \mathbb{R}^n$. Suppose $B\mathbf{v} = \mathbf{0}$.
      Then $(AB)\mathbf{v} = A(B\mathbf{v}) = A\mathbf{0} = \mathbf{0}$.
      Since AB is non-singular and $(AB)\mathbf{v} = \mathbf{0}$, we have $\mathbf{v} = \mathbf{0}$.
      It follows that B is non-singular.

    Now, by Theorem (B), the matrix B is invertible, with matrix inverse B^{-1}. Note that B^{-1} is invertible.

    Note that $A = AI_n = A(BB^{-1}) = (AB)B^{-1}$.

    Since AB is invertible and B^{-1} is invertible, A is also invertible according to Lemma (β).

    Now, by Theorem (B), the matrix A is non-singular.

    Remark. We may combine Lemma (β') and Lemma (η) to obtain Theorem (D).

16. **Theorem (D).**

    Suppose A, B are $(n \times n)$-square matrices.

    Then the statements below are logically equivalent:

    ($\sharp$) *Each of A, B is non-singular and invertible.*

    ($\flat$) *The product AB is non-singular and invertible.*

17. **Corollary to Theorem (D).**

    Suppose $B_1, B_2, \cdots, B_k$ are $(n \times n)$-square matrices.

    Then the statements below are logically equivalent:

    ($\sharp\sharp$) *Each of $B_1, B_2, \cdots, B_k$ is non-singular and invertible.*

    ($\flat\flat$) *The product $B_1B_2\cdots B_k$ is non-singular and invertible.*

    Proof of Corollary to Theorem (D). Apply Theorem (D) and mathematical induction.