

1. **How to re-formulate the notion of non-singularity in terms of row operations and reduced row-echelon forms?**

Recall the definition for the notion of *non-singularity*:

Suppose A is a square matrix. Then A is non-singular if and only if the trivial solution is the only solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

So, from the definition, when A is indeed a square matrix and A' is the reduced row-echelon form which is row-equivalent to A , we will have the mutually exclusive scenarios ($\dagger$), ($\dagger'$) below, dependent on whether A' is the identity matrix or not:—

- ($\dagger$) Suppose $A' = I_n$. Then the trivial solution is the only solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. Therefore A is non-singular.
- ($\dagger'$) Suppose $A' \neq I_n$. Then A' is singular. Therefore there is a non-trivial solution for the homogeneous system $\mathcal{LS}(A', \mathbf{0})$, which is also a non-trivial solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. Hence A is singular.

Summarizing the discussion above, we have obtained a re-formulation for the notion of non-singularity in terms of reduced row-echelon forms.

2. **Lemma (4).**

Let A be a square matrix.

A is non-singular if and only if the reduced row-echelon form which is row-equivalent to A is given by the identity matrix.

Remark. In the light of the Lemma (2), we may state Lemma (4) in this way:

Let A be a square matrix. The following statements are logically equivalent:

- (a) A is non-singular.
- (b) A is row equivalent to I_n .

3. **Example ($\star$), as illustrations for Lemma (4).**

(a) Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix}$.

We find the reduced row-echelon form A' which is row-equivalent to A , say, through Gaussian elimination:

$$(\alpha): \quad A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{17}} C_{18} \xrightarrow{\rho_{18}} C_{19} = A'$$

in which the ρ_k 's are given by:

k	ρ_k	k	ρ_k	k	ρ_k
1	$2R_1 + R_2$	7	$1R_1 + R_5$	13	$-1R_4 + R_1$
2	$-1R_1 + R_3$	8	$1R_3 + R_4$	14	$-1R_4 + R_2$
3	$2R_1 + R_4$	9	$-2R_3 + R_5$	15	$1R_4 + R_3$
4	$1R_1 + R_5$	10	$-2R_2 + R_1$	16	$-2R_5 + R_1$
5	$1R_2 + R_3$	11	$3R_3 + R_1$	17	$1R_5 + R_2$
6	$-1R_2 + R_4$	12	$-2R_3 + R_2$	18	$-1R_5 + R_3$

It happens that $A' = I_5$. Then we may conclude that A is non-singular.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

We find the reduced row-echelon form A' which is row-equivalent to A , say, through Gaussian elimination:

$$(\alpha): \quad A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{11}} C_{12} \xrightarrow{\rho_{12}} C_{13} = A'$$

in which the ρ_k 's are given by:

k	ρ_k	k	ρ_k	k	ρ_k
1	$-1R_1 + R_2$	5	$R_3 \leftrightarrow R_4$	9	$-1R_3 + R_2$
2	$-1R_1 + R_3$	6	$1R_3 + R_4$	10	$1R_4 + R_1$
3	$R_2 \leftrightarrow R_4$	7	$(1/3)R_4$	11	$1R_4 + R_2$
4	$1R_2 + R_4$	8	$-1R_2 + R_1$	12	$-2R_4 + R_3$

It happens that $A' = I_4$. Then we may conclude that A is non-singular.

(c) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 4 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$.

We find the reduced row-echelon form A' which is row-equivalent to A , say, through Gaussian elimination:

$$(\alpha) : \quad A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_8} C_9 \xrightarrow{\rho_9} C_{10} = A'$$

in which the ρ_k 's are given by:

$$\begin{array}{c|c} k & \rho_k \\ \hline 1 & -1R_1 + R_2 \\ 2 & -3R_1 + R_3 \\ 3 & -2R_1 + R_4 \end{array} \parallel \begin{array}{c|c} k & \rho_k \\ \hline 4 & R_2 \leftrightarrow R_3 \\ 5 & 1R_2 + R_3 \\ 6 & -1R_3 \end{array} \parallel \begin{array}{c|c} k & \rho_k \\ \hline 7 & 1R_3 + R_4 \\ 8 & -1R_2 + R_1 \\ 9 & -1R_3 + R_2 \end{array}$$

It happens that $A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then we may conclude that A is singular.

Remark. In fact, C_8 is a row-echelon form with an entire row of zeros. We can stop there and conclude that A is singular. (Why?)

4. How to re-formulate the notion of non-singularity in terms of matrix multiplication?

Let A be an $(n \times n)$ -square matrix, and A' be the reduced row-echelon form which is row-equivalent to A . We make some observations with the help of the 'dictionary' between row operations and row-operation matrices:

(a) For such matrices A, A' , we have some sequence of row operations joining A to A' :

$$A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{p-2}} C_{p-1} \xrightarrow{\rho_{p-1}} C_p = A'.$$

According to the 'dictionary' between row operations and row-operation matrices, for each j , there is some (unique) $(n \times n)$ -square matrix H_j , namely, the row operation matrix for ρ_j , such that $C_{j+1} = H_j C_j$.

Then $A' = C_p = H_{p-1} C_{p-1} = H_{p-1} H_{p-2} C_{p-2} = \cdots = H_{p-1} \cdots H_2 H_1 A$.

(b) Now further suppose A is non-singular. Then, by Lemma (4), $A' = I_n$.

Therefore there exist some $(n \times n)$ -square matrix H , namely, $H = H_{p-1} \cdots H_2 H_1$, such that $HA = I_n$.

We summarize the discovery in the above discussion in the form of Lemma (5).

5. Lemma (5). (Converse of Lemma (2).)

Let A be an $(n \times n)$ -square matrix.

Suppose A is non-singular. Then there exists some $(n \times n)$ -square matrix H such that $HA = I_n$.

Remark. Why do we call Lemma (5) a 'converse of Lemma (2)'?

It is because Lemma (2) reads as:

Let A be an $(n \times n)$ -square matrix.

Suppose there exists some $(n \times n)$ -square matrix H such that $HA = I_n$. Then A is non-singular.

Combining Lemma (1), Lemma (2), Lemma (4) and Lemma (5), we obtain Theorem (A).

6. Theorem (A). (Re-formulation of non-singularity in terms of row operations, reduced row-echelon forms and matrix multiplication.)

Let A be an $(n \times n)$ -square matrix. The statements below are logically equivalent:

- (a) A is non-singular.
- (b) For any vector $\mathbf{v}$ in $\mathbb{R}^n$, if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.
- (c) The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.
- (d) A is row-equivalent to I_n .
- (e) There exists some $(n \times n)$ -square matrix H such that $HA = I_n$.

7. Towards the notion of invertibility.

Now indeed suppose A is a non-singular $(n \times n)$ -square matrix, which, according to Theorem (A), will satisfy $HA = I_n$ for some appropriate $(n \times n)$ -square matrix H .

(a) **Question.** How to write down such a matrix H explicitly?

Answer. According to our discussion leading up to Lemma (5), such a matrix H can be obtained as the product $H = H_{p-1} \cdots H_2 H_1$ in which each H_j is the row operation matrix corresponding to the row operation ρ_j in some sequence of row operations joining A to I_n :

$$(\alpha): \quad A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{p-2}} C_{p-1} \xrightarrow{\rho_{p-1}} C_p = I_n.$$

(b) **Further question.** But can we obtain H without multiplying so many matrices together?

Answer to further question. Yes. How?

Write $H = H_{p-1} \cdots H_2 H_1 I_n$ (as a sleight of hand), and think of how to interpret this product in terms of row operations, according to the ‘dictionary’ between row operations and row operation matrices.

H_1 is the resultant of the application of the row operation ρ_1 on I_n .

$H_2 H_1$ is the resultant of the application of the row operation ρ_2 on H_1 .

$H_3 H_2 H_1$ is the resultant of the application of the row operation ρ_3 on $H_2 H_1$. So forth and so on.

H is therefore the resultant of the sequence of row operations

$$(\beta): \quad I_n \xrightarrow{\rho_1} H_1 \xrightarrow{\rho_2} H_2 H_1 \xrightarrow{\rho_3} \cdots \xrightarrow{\rho_{p-2}} H_{p-2} \cdots H_2 H_1 \xrightarrow{\rho_{p-1}} H_{p-1} \cdots H_2 H_1 = H.$$

(c) **Bonus.** Again look at the sequence

$$(\beta): \quad I_n \xrightarrow{\rho_1} H_1 \xrightarrow{\rho_2} H_2 H_1 \xrightarrow{\rho_3} \cdots \xrightarrow{\rho_{p-2}} H_{p-2} \cdots H_2 H_1 \xrightarrow{\rho_{p-1}} H_{p-1} \cdots H_2 H_1 = H.$$

This tells us immediately that:

- I_n is row-equivalent to the $(n \times n)$ -square matrix H , and hence
- the $(n \times n)$ -square matrix H is also row-equivalent to I_n .

Then, by Theorem (A), the statements below all hold immediately and simultaneously for this matrix H :

- H is non-singular.
- For any vector $\mathbf{u}$ in $\mathbb{R}^n$, if $H\mathbf{u} = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$.
- The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(H, \mathbf{0})$.
- There exists some $(n \times n)$ -square matrix G such that $GH = I_n$.

So now we know that $HA = I_n$, and some matrix G satisfies $GH = I_n$.

(d) **Question.** What is the matrix G ?

Answer. The matrix G can be nothing but A itself. Justification:

For such matrices A, G, H , we have $G = GI_n = G(HA) = (GH)A = I_n A = A$.

Extra bonus. Therefore, for the same matrices A, H , it happens not only the equality $HA = I_n$ holds, but also the equality $AH = I_n$ holds.

We have obtained something unexpected discovery from the ‘practical problem’ of computing the matrix H which satisfies $HA = I_n$ for the non-singular matrix A . We formulate this discovery as Lemma (6).

8. Lemma (6).

Let A be an $(n \times n)$ -square matrix.

Suppose A non-singular.

Then there exists some $(n \times n)$ -square matrix H such that H is non-singular, $HA = I_n$ and $AH = I_n$.

Remark. The converse of Lemma (6) is the statement ($\sharp$):

($\sharp$) Let A be an $(n \times n)$ -square matrix.

Suppose there exists some $(n \times n)$ -square matrix H such that H is non-singular, $HA = I_n$ and $AH = I_n$.

Then A non-singular.

The statement ($\sharp$) is certainly true, by virtue of Theorem (A).

But how about statement (b) below?

(b) Let A be an $(n \times n)$ -square matrix.

Suppose there exists some $(n \times n)$ -square matrix H such that $AH = I_n$.

Then A non-singular.

We are not so sure at this point, as we are assuming ‘less’ in Statement (b) then in Statement (#).

It will transpire that Statement (b) is true as well, after more work is done.

9. **Example (★★), as an illustration for Lemma (6).**

Recall Example (★). Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix}$.

We have the sequence of row operations

$$(\alpha) : A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{17}} C_{18} \xrightarrow{\rho_{18}} C_{19} = A',$$

in which the row operations ρ_k and the corresponding row-operation matrices H_k are given by:

k	ρ_k	H_k	k	ρ_k	H_k	k	ρ_k	H_k
1	$2R_1 + R_2$	$I_5 + 2E_{2,1}^{5,5}$	7	$1R_1 + R_5$	$I_5 + E_{5,2}^{5,5}$	13	$-1R_4 + R_1$	$I_5 - 1E_{1,4}^{5,5}$
2	$-1R_1 + R_3$	$I_5 - E_{3,1}^{5,5}$	8	$1R_3 + R_4$	$I_5 + E_{4,3}^{5,5}$	14	$-1R_4 + R_2$	$I_5 - E_{2,4}^{5,5}$
3	$2R_1 + R_4$	$I_5 + 2E_{4,1}^{5,5}$	9	$-2R_3 + R_5$	$I_5 - 2E_{5,3}^{5,5}$	15	$1R_4 + R_3$	$I_5 + E_{3,4}^{5,5}$
4	$1R_1 + R_5$	$I_5 + E_{5,1}^{5,5}$	10	$-2R_2 + R_1$	$I_5 - 2E_{1,2}^{5,5}$	16	$-2R_5 + R_1$	$I_5 - 2E_{1,5}^{5,5}$
5	$1R_2 + R_3$	$I_5 + E_{3,2}^{5,5}$	11	$3R_3 + R_1$	$I_5 + 3E_{1,3}^{5,5}$	17	$1R_5 + R_2$	$I_5 + E_{2,5}^{5,5}$
6	$-1R_2 + R_4$	$I_5 - E_{4,2}^{5,5}$	12	$-2R_3 + R_2$	$I_5 - 2E_{2,3}^{5,5}$	18	$-1R_5 + R_3$	$I_5 - E_{3,5}^{5,5}$

It happens that $I_5 = A' = C_{19} = H_{18}C_{18} = H_{18} \cdots H_2H_1A$.

The matrix $H = H_{18}C_{18} = H_{18} \cdots H_2H_1$ is the resultant of the application of the row operations $\rho_1, \rho_2, \dots, \rho_{18}$ on I_5 , and is explicitly given by

$$H = \begin{bmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{bmatrix}.$$

It so happens that H is non-singular, $HA = I_5$ and $AH = I_5$.