1. **Definition. (Null space of a matrix.)**

   Let A be an $(m \times n)$-matrix.

   (a) *The system of linear equations $\mathcal{LS}(A, \mathbf{0})$ is called the homogeneous system with coefficient matrix A.*

   (b) *The solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ is called the null space of A. It is denoted by $\mathcal{N}(A)$.*

   Remark. First of all, recall that

   '$\mathbf{x} = \mathbf{u}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$ if and only if $A\mathbf{u} = \mathbf{0}$.

   Using as 'selection criterion' the equality '$A\mathbf{x} = \mathbf{0}$', we may present the null space of A as a set constructed using the method of specification:

   - Those vectors in $\mathbb{R}^n$ which, upon substitution into the '$\mathbf{x}$' in this 'selection criterion' result in an equality, will be 'collected'.
   - Those vectors in $\mathbb{R}^n$ which, upon substitution into the '$\mathbf{x}$' in this 'selection' do not result in an equality, will be 'discarded'.

   Hence the null space of A is the set $\{\mathbf{u} \in \mathbb{R}^n : A\mathbf{u} = \mathbf{0}\}$.

   Further remark. How to use the various versions of the definitions?

   Always remember, whenever $\mathbf{v} \in \mathbb{R}^n$, the statements below mean the same thing:

   (a) $\mathbf{v} \in \mathcal{N}(A)$.

   (b) $A\mathbf{v} = \mathbf{0}$.

   (c) *'$\mathbf{x} = \mathbf{v}$' is a solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ with unknown $\mathbf{x}$.*

   To determine $\mathcal{N}(A)$ is the same as giving an 'explicit' description of the solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ through set language, in terms of (hopefully just a few) solutions of the system. That amounts to finding all solutions of $\mathcal{LS}(A, \mathbf{0})$.

2. **Example ($\star$).**

   Determine the null space of the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

   explicitly (in terms of concrete vectors in $\mathbb{R}^7$).

   (a) First determine the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations, say, Gaussian elimination, to the augmented matrix representation of $\mathcal{LS}(A, \mathbf{0})$:

$$[A|\mathbf{0}] \longrightarrow \cdots\cdots\cdots \longrightarrow [A'|\mathbf{0}]$$

   We find that

$$A' = \begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (b) $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\left\{ \begin{array}{ccccccccccccccc} x_1 & + & 4x_2 & & & & & + & 2x_5 & + & x_6 & - & 3x_7 & = & 0 \\ & & & & x_3 & & & + & x_5 & - & 3x_6 & + & 5x_7 & = & 0 \\ & & & & & & x_4 & + & 2x_5 & - & 6x_6 & + & 6x_7 & = & 0 \\ & & & & & & & & & & & & 0 & = & 0 \end{array} \right.$$

   The solutions of $\mathcal{LS}(A', \mathbf{0})$, and hence of $\mathcal{LS}(A, \mathbf{0})$, are given by $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$, where c_1, c_2, c_3, c_4 are arbitrary numbers, in which $\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   This amounts to saying that for any $\mathbf{v} \in \mathbb{R}^7$, '$\mathbf{x} = \mathbf{v}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$ if and only if there exist some $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$.

(c) We now apply the method of specification to present the solution set of $\mathcal{LS}(A, \mathbf{0})$ explicitly in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, through the 'selection criterion'

(†) ' *there exist some* $c_1, c_2, c_3, c_4 \in \mathbb{R}$ *such that* $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$'

in which $\mathbf{y}$ is the 'variable'.

How does the method work? Remember:

- Those and only those vectors in $\mathbb{R}^7$ which upon substitution into the symbol $\mathbf{y}$ in (†) turn it into a true statement will be collected.
- The others will be 'discarded'.

So $\mathcal{N}(A)$ is the set

$$\left\{\mathbf{y} \in \mathbb{R}^7 : \begin{array}{l} \text{there exist some } c_1, c_2, c_3, c_4 \in \mathbb{R} \\ \text{such that } \mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \end{array}\right\}$$

(As shorthand we present $\mathcal{N}(A)$ as $\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}$.)

(d) **Comment on the presentation of the manipulations.**

During the manipulation

$$[A|\mathbf{0}] \longrightarrow \cdots\cdots\cdots \longrightarrow [A'|\mathbf{0}]$$

we observe that the last column in every matrix in this sequence stays '$\mathbf{0}$'. This is expected: no matter which row-operation is applied on the zero vector, it only convert the zero vector to itself.

Hence we can actually save time (and ink) by omitting the $\mathbf{0}$'s throughout, and simply write

$$A \longrightarrow \cdots\cdots\cdots \longrightarrow A'$$

provided we remember we are apply row operations on the coefficient matrices of various homogeneous system.

3. **Examples on determining null space explicitly.**

(a) *Determine the null space of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$$

Determine the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 2 & 6 & 5 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \\ &\xrightarrow{-1R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A' \end{aligned}$$

The null space $\mathcal{N}(A)$ of the matrix A is the solution set of $\mathcal{LS}(A, \mathbf{0})$, and hence is the solution set of $\mathcal{LS}(A', \mathbf{0})$ as well. Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\left\{ \begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \\ x_3 & = & 0 \end{array} \right.$$

The only solution of $\mathcal{LS}(A, \mathbf{0})$ is given by $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Hence $\mathcal{N}(A)$ is the set $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(b) *Determine the null space of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$$

Determine the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix} \\ &\xrightarrow{-1R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-1R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} = A' \end{aligned}$$

The null space $\mathcal{N}(A)$ of the matrix A is the solution set of $\mathcal{LS}(A,\, \mathbf{0})$, and hence is the solution set of $\mathcal{LS}(A',\, \mathbf{0})$ as well. Note that $\mathcal{LS}(A',\, \mathbf{0})$ reads:

$$\left\{\begin{array}{ccccccc} x_1 & & & + & 2x_4 & = & 0 \\ & x_2 & & - & 3x_4 & = & 0 \\ & & x_3 & + & 4x_4 & = & 0 \end{array}\right.$$

The solutions of $\mathcal{LS}(A,\, \mathbf{0})$ are given by $\mathbf{x} = t\mathbf{u}$, where t is an arbitrary number, in which $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$.

Hence $\mathcal{N}(A)$ is the set $\{\, t\mathbf{u} \mid t \in \mathbb{R}\}$.

(c) *Determine the null space of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$$

Determine the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & -5 & 5 & -10 & -20 \end{bmatrix}$$

$$\xrightarrow{-1R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & -5 & 5 & -10 & -20 \end{bmatrix} \xrightarrow{5R_2+R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_6+R_1} \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

The null space $\mathcal{N}(A)$ of the matrix A is the solution set of $\mathcal{LS}(A,\, \mathbf{0})$, and hence is the solution set of $\mathcal{LS}(A',\, \mathbf{0})$ as well. Note that $\mathcal{LS}(A',\, \mathbf{0})$ reads:

$$\left\{\begin{array}{ccccccccc} x_1 & & + & 2x_3 & - & 3x_4 & - & x_5 & = & 0 \\ & x_2 & - & x_3 & + & 2x_4 & + & 4x_5 & = & 0 \\ & & & & & & & 0 & = & 0 \end{array}\right.$$

The solutions of $\mathcal{LS}(A,\, \mathbf{0})$ are given by $\mathbf{x} = c_1\mathbf{u}_1 + c\mathbf{u}_2 + c_3\mathbf{u}_3$, where c_1, c_2, c_3 are arbitrary numbers, in which $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence $\mathcal{N}(A)$ is the set $\{\, c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$.

4. **Further consideration on Example ($\star$).**

Let

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

Recall that $\mathcal{N}(A)$ *is the set* $\{\, c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}$, *in which* $\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$,

$\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(a) **Further question.**

What is so special about $\mathcal{N}(A)$, regarding its 'algebraic structure'?

Answer to further question.

The statements below hold:

(1) $\mathbf{0} \in \mathcal{N}(A)$.

(2) *For any* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$, *if* $\mathbf{u}, \mathbf{v} \in \mathcal{N}(A)$ *then* $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

(3) *For any* $\mathbf{v} \in \mathbb{R}^7$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{v} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{v} \in \mathcal{N}(A)$.

(4) *For any* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$, *for any* $\alpha, \beta \in \mathbb{R}$, *if* $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{v} + \beta\mathbf{w} \in \mathcal{N}(A)$.

(b) **Justification of (1), (2), (3) in answer to further question.**

To apply what we see about $\mathcal{N}(A)$ in concrete terms?

(1) Note that $\mathbf{0} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + 0 \cdot \mathbf{u}_3 + 0 \cdot \mathbf{u}_4$, and $0 \in \mathbb{R}$. Then $\mathbf{0} \in \mathcal{N}(A)$.

(2) Pick any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$.
Suppose $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$. Then there exist some $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4 \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$ and $\mathbf{w} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + d_3\mathbf{u}_3 + d_4\mathbf{u}_4$.
Then $\mathbf{v}+\mathbf{w} = \cdots = (c_1+d_1)\mathbf{u}_1+(c_2+d_2)\mathbf{u}_2+(c_3+d_3)\mathbf{u}_3+(c_4+d_4)\mathbf{u}_4$, and $c_1+d_1, c_2+d_2, c_3+d_3, c_4+d_4 \in \mathbb{R}$.
Therefore $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

(3) Pick any $\mathbf{v} \in \mathbb{R}^7$. Pick any $\alpha \in \mathbb{R}$.
Suppose $\mathbf{v} \in \mathcal{N}(A)$. Then there exists some $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$.
Then $\alpha\mathbf{v} = \cdots = (\alpha c_1)\mathbf{u}_1 + (\alpha c_2)\mathbf{u}_2 + (\alpha c_3)\mathbf{u}_3 + (\alpha c_4)\mathbf{u}_4$, and $\alpha c_1, \alpha c_2, \alpha c_3, \alpha c_4 \in \mathbb{R}$.
Therefore $\alpha\mathbf{v} \in \mathcal{N}(A)$.

(c) **Another justification of (1), (2), (3) in answer to further question.**

To apply the definition of null space? (This is a better method.)

(1) Note that $A\mathbf{0} = \mathbf{0}$. Then $\mathbf{0} \in \mathcal{N}(A)$.

(2) Pick any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$.
Suppose $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$. Then $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$.
Therefore $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
Hence $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

(3) Pick any $\mathbf{v} \in \mathbb{R}^7$. Pick any $\alpha \in \mathbb{R}$.
Suppose $\mathbf{v} \in \mathcal{N}(A)$. Then $A\mathbf{v} = \mathbf{0}$.
Therefore $A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha \cdot \mathbf{0} = \mathbf{0}$.
Hence $\alpha\mathbf{v} \in \mathcal{N}(A)$.

Remark. This 'second justification' of (1), (2), (3) is superior to the 'first', in the sense that almost nothing about explicit features of A, apart from the fact that it has 7 columns, is involved in the mathematical argument. We may wonder the mathematical reasoning in this 'second justification' may work when A is replaced by a general matrix. It turns out to be the case.

5. **Theorem (1). (Null space of a matrix as a 'subspace'.)**

Let A be an $(m \times n)$-matrix. The statement below hold:

(1) $\mathbf{0} \in \mathcal{N}(A)$.

(2) *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *if* $\mathbf{u}, \mathbf{v} \in \mathcal{N}(A)$ *then* $\mathbf{u} + \mathbf{v} \in \mathcal{N}(A)$.

(3) *For any* $\mathbf{u} \in \mathbb{R}^n$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{u} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{u} \in \mathcal{N}(A)$.

(4) *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *for any* $\alpha, \beta \in \mathbb{R}$, *if* $\mathbf{u}, \mathbf{v} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{u} + \beta\mathbf{v} \in \mathcal{N}(A)$.

Proof. Exercise. (Extract what we did in the study of Example ($\star$).)

Remark. We can further deduce that

For any $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathbb{R}^n$, *for any* $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$, *if* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathcal{N}(A)$ *then* $\alpha\mathbf{u}_1 + \alpha\mathbf{u}_2 + \cdots + \alpha\mathbf{u}_k \in \mathcal{N}(A)$.

In plain words (and in terms of the notion of *linear combinations*, to be introduced later), this amounts to saying:

Every linear combination of vectors in $\mathcal{N}(A)$ is a vector in $\mathcal{N}(A)$.

6. **Reformulation of Theorem (1) in terms of homogeneous systems.**

Let A be an $(m \times n)$-matrix. The statement below hold:

(1) '$\mathbf{x} = \mathbf{0}$' *is a solution of* $\mathcal{LS}(A, \mathbf{0})$.

(2) *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Suppose* '$\mathbf{x} = \mathbf{u}$', '$\mathbf{x} = \mathbf{v}$' *are solutions of* $\mathcal{LS}(A, \mathbf{0})$. *Then* '$\mathbf{x} = \mathbf{u} + \mathbf{v}$' *is a solution of* $\mathcal{LS}(A, \mathbf{0})$.

(3) *Let* $\mathbf{u} \in \mathbb{R}^n$. *Let* $\alpha \in \mathbb{R}$. *Suppose* '$\mathbf{x} = \mathbf{u}$' *is a solution of* $\mathcal{LS}(A, \mathbf{0})$. *Then* '$\mathbf{x} = \alpha\mathbf{u}$' *is a solution of* $\mathcal{LS}(A, \mathbf{0})$.

(4) *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Let* $\alpha, \beta \in \mathbb{R}$. *Suppose* '$\mathbf{x} = \mathbf{u}$', '$\mathbf{x} = \mathbf{v}$' *are solutions of* $\mathcal{LS}(A, \mathbf{0})$. *Then* '$\mathbf{x} = \alpha\mathbf{u} + \beta\mathbf{v}$' *is a solution of* $\mathcal{LS}(A, \mathbf{0})$.