0. This handout is meant to be a continuation of the handout *Examples of simple proofs in linear algebra.*

   Here we give a brief description on various notions in mathematical logic and reasoning, mostly through examples, which will suffice for use in this course.

   (MATH/BMED students will have to learn much more and in greater depth on the same matter in their next MATH course for level-2000 proof-type MATH courses.)

1. **Conditional statement and its format.**

   Many statements in linear algebra can be formulated in this form of a 'three-sentence passage':

   ($\star$) '**Let** the object *so-and-so* be amongst the objects *blah-blah-blah*. **Suppose** the object *so-and-so* possesses the property *bleh-bleh-bleh*. **Then** the object *so-and-so* possesses the property *blih-blih-blih*.'

   Example (A) (from the handout *Examples of simple proofs in linear algebra*):

   (a) **Let** A be an $(n \times n)$-square matrix. **Suppose** A is symmetric and A is skew-symmetric. **Then** $A = \mathcal{O}_{n\times n}$.

   (b) **Let** A, B, C be $(n \times n)$-square matrices. **Suppose** each of B, C is a matrix inverse of A. **Then** $B = C$.

   (c) **Let** A be an $(n \times n)$-square matrix. **Suppose** $A - I_n$ is idempotent. **Then** A is invertible.

   (d) **Let** A be an $(n \times n)$-square matrix. **Suppose** A is idempotent, and A is not the identity matrix. **Then** there exists some non-zero vector $\mathbf{v}$ in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$.

   (e) **Let** A be an $(n \times n)$-square matrix. **Suppose** A is not the zero matrix and A is nilpotent. **Then** $I_n - A$ is invertible, and there is some positive integer k so that $I_n + A + A^2 + \cdots + A^k$ is a matrix inverse of $I_n - A$.

   (f) **Let** A, B be $(n \times n)$-square matrices. **Suppose** $[A, B] = \mathcal{O}_{n\times n}$. **Then** for any positive integer p, $A^k B = BA^k$.

   (g) **Let** A be an $(n \times n)$-square matrix. **Suppose** A is nilpotent. **Then** A is not invertible.

   (h) **Let** A be an $(n \times n)$-square matrix. **Suppose** A is idempotent and A is not the zero matrix. **Then** A is not nilpotent.

   Such a statement is called a conditional statement in mathematics.

   - The information '*so-and-so* is amongst *blah-blah-blah*' and '*so-and-so* possesses bleh-bleh-bleh' is collectively referred to as 'assumption in the statement'.
   - The information '*so-and-so* possesses blih-blih-blih' is referred to as 'conclusion in the statement'.
   - Very often the most important portion of the assumption (which we hope will lead to the conclusion) is placed in between the bold-type word '**suppose**' and the bold-type word '**then**'. (In between the words '**let**', '**suppose**' we only lay out the most general information on where we may 'locate' the 'type of objects' under consideration throughout the statement.)

   Remarks.

   (a) When the assumption in a conditional statement is of the form '*bleh-bleh-bleh and bleh-bleh-bleh and ...*' and so looks lengthy, we may agree to split the assumption into shorter sentences. Example:

   The statement

   'Let A be an $(n \times n)$-square matrix. Suppose A is idempotent, and A is not the identity matrix. Then there exists some non-zero vector $\mathbf{v}$ in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$.'

   can be re-written as:

   'Let A be an $(n \times n)$-square matrix. Suppose A is idempotent. Further suppose A is not the identity matrix. Then there exists some non-zero vector $\mathbf{v}$ in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$.'

   (b) When the conclusion in a conditional statement if of the form '*blih-blih-blih and blih-blih-blih and ...*' and so looks lengthy, we may also agree to split the conclusion into shorter sentences. Example:

   The statement

   'Let A be an $(n \times n)$-square matrix. Suppose A is not the zero matrix and A is nilpotent. Then $I_n - A$ is invertible, and there is some positive integer k so that $I_n + A + A^2 + \cdots + A^k$ is a matrix inverse of $I_n - A$.'

   can be re-written as:

   'Let A be an $(n \times n)$-square matrix. Suppose A is not the zero matrix and A is nilpotent. Then $I_n - A$ is invertible. Moreover, there is some positive integer k so that $I_n + A + A^2 + \cdots + A^k$ is a matrix inverse of $I_n - A$.'

2. **'Compact' presentation of conditional statements.**

   The conditional statement

   ($\star$) '**Let** the object *so-and-so* be amongst the objects *blah-blah-blah*. **Suppose** the object *so-and-so* possesses the property *bleh-bleh-bleh*. **Then** the object *so-and-so* possesses the property *blih-blih-blih*.'

   can be presented in a 'compact' 'one-sentence' form:

($\star'$) '**For any** object *so-and-so* amongst the objects *blah-blah-blah*, **if** the object *so-and-so* possesses the property *bleh-bleh-bleh*, **then** the object *so-and-so* possesses the property *blih-blih-blih*.'

Example (A'). The statements listed under Example (A) can be respectively presented as:

(a) **For any** $(n \times n)$-square matrix A, **if** A is symmetric and A is skew-symmetric **then** $A = \mathcal{O}_{n\times n}$.

(b) **For any** $(n \times n)$-square matrices A, B, C, **if** each of B, C is a matrix inverse of A, **then** $B = C$.

(c) **For any** $(n \times n)$-square matrix A, **if** $A - I_n$ is idempotent, **then** A is invertible.

(d) **For any** $(n \times n)$-square matrix A, **if** A is idempotent, and A is not the identity matrix, **then** there exists some non-zero vector $\mathbf{v}$ in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$.

(e) **For any** $(n \times n)$-square matrix A, **if** A is not the zero matrix and A is nilpotent, **then** $I_n - A$ is invertible, and there is some positive integer k so that $I_n + A + A^2 + \cdots + A^k$ is a matrix inverse of $I_n - A$.

(f) **For any** $(n \times n)$-square matrices A, B, **if** $[A, B] = \mathcal{O}_{n\times n}$, **then** for any positive integer p, $A^k B = BA^k$.

(g) **For any** $(n \times n)$-square matrix A, **if** A is nilpotent, **then** A is not invertible.

(h) **For any** $(n \times n)$-square matrix A, **if** A is idempotent and A is not the zero matrix, **then** A is not nilpotent.

3. An arbitrary conditional statement may be true, or false.

   (a) When we claim that a conditional statement is true, we can justify this claim by giving a proof for the conditional statement.
   Examples of such work (in proving conditional statements) can be found in the handout *Examples of simple proofs in linear algebra*.

   (b) When we claim that a conditional statement is false, we can justify this claim by giving a dis-proof against the conditional statement.

4. **Dis-proving conditional statements.**

   Imagine we want to dis-prove the conditional statement

   ($\star$) '**Let** the object *so-and-so* be amongst the objects *blah-blah-blah*. **Suppose** the object *so-and-so* possesses the property *bleh-bleh-bleh*. **Then** the object *so-and-so* possesses the property *blih-blih-blih*.'

   This amounts to proving the 'existence statement' which reads:

   ($\sim\star$) '**There exists some** object *so-and-so* amongst the objects *blah-blah-blah* such that the object *so-and-so* possesses the property *bleh-bleh-bleh* **and** the object *so-and-so* does **not** possess the property *blih-blih-blih*.'

   In practice, we often proceed as described below to construct the argument for ($\sim\star$):

   - **Step (0).** (This is the preparation for the argument, and does not count as part of the argument.)
     Conceive through whatever means appropriate (say, by roughwork calculations, by an educated guess, by trial-and-error, or by a combination of all these) a candidate 'concrete' object *so-and-so* which we believe will be amongst the objects *blah-blah-blah* and will possess the property *bleh-bleh-bleh* and will not possess the property *blih-blih-blih*.
   - **Step (1).** (This is the beginning of the argument.)
     Name the candidate 'concrete' object.
   - **Step (2).**
     Confirm, by giving appropriate justifications if necessary, that the candidate 'concrete' object named in Step (1) is indeed amongst the objects *blah-blah-blah*.
   - **Step (3).**
     Confirm, by giving appropriate justifications if necessary, that the candidate 'concrete' object named in Step (1) indeed possesses the property *bleh-bleh-bleh*.
   - **Step (4).**
     Confirm, by giving appropriate justifications if necessary, that the candidate 'concrete' object named in Step (1) indeed does not possess the property *blih-blih-blih*.

   The order of Step (2), Step (3), Step (4) may be permuted.

   The 'concrete' object *so-and-so* named in Step (1) is called a counter-example against the conditional statement ($\star$).

5. **Examples of dis-proofs against conditional statements.**

   (a) We want to dis-prove the conditional statement

   (P) 'Let A be an $(n \times n)$-square matrix. Suppose A is invertible. Then $A - I_n$ is idempotent.'

   This amount to proving the statement

   ($\sim P$) 'There exists some $(n \times n)$-square matrix A such that A is invertible and $A - I_n$ is not idempotent.'

   Below is the argument for ($\sim P$) (and hence the argument against (P)):

[Preparation. By trial-and-error (starting with (2×2)-matrices which have as many entries being 0 or 1), we see that when $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, it seems that A is invertible and $A - I_n$ is idempotent.]

Take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Note that A is a (2×2)-square matrix.

We have $A^2 = \cdots = I_2$. Then A is invertible, with matrix inverse being A itself.

Note that $A - I_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

Then $(A - I_2)^2 = \cdots = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \neq A - I_2$.

Therefore $A - I_2$ is not idempotent.

(b) We want to dis-prove the conditional statement

(P) 'Let A be an $(n \times n)$-square matrix. Suppose A is not invertible. Then A is nilpotent.'

This amount to proving the statement

($\sim P$) 'There exists some $(n \times n)$-square matrix A such that A is not invertible and A is not nilpotent.'

Below is the argument for ($\sim P$) (and hence the argument against (P)):

[Preparation. By trial-and-error (starting with (2×2)-matrices which have as many entries being 0 or 1), we see that when $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, it seems that A is not invertible and A is not nilpotent.]

Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Note that A is a (2×2)-square matrix.

We have $A^2 = \cdots = A$. Then, for each positive integer p, we have $A^p = \cdots = A^2 = A \neq \mathcal{O}_{2\times 2}$. Therefore A is not nilpotent.

We verify that for each (2×2)-matrix B, $AB \neq I_2$:

- Suppose B is a (2×2)-matrix whose (i, j)-th entry is b_{ij}.
  Then $AB = \cdots = \begin{bmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ 0 & 0 \end{bmatrix} \neq I_2$.
  It follows that A is not invertible.

6. **Converse of a conditional statement.**

Consider the conditional statement

($\star$) '**Let** the object *so-and-so* be amongst the objects *blah-blah-blah*. **Suppose** the object *so-and-so* possesses the property *bleh-bleh-bleh*. **Then** the object *so-and-so* possesses the property *blih-blih-blih*.'

When we interchange the positions of '*bleh-bleh-bleh*' and '*blih-blih-blih*' inside ($star$), we obtain another conditional statement, which reads:

($\widehat{\star}$) '**Let** the object *so-and-so* be amongst the objects *blah-blah-blah*. **Suppose** the object *so-and-so* possesses the property *blih-blih-blih*. **Then** the object *so-and-so* possesses the property *bleh-bleh-bleh*.'

The conditional statement ($\widehat{\star}$) is called the converse of the conditional statement ($\star$).

Note that ($\star$) itself is the converse of ($\widehat{\star}$).

Their corresponding 'compact' 'one-sentence' forms read respectively as:

($\star'$) '**For any** object *so-and-so* amongst the objects *blah-blah-blah*, **if** the object *so-and-so* possesses the property *bleh-bleh-bleh*, **then** the object *so-and-so* possesses the property *blih-blih-blih*.'

($\widehat{\star'}$) '**For any** object *so-and-so* amongst the objects *blah-blah-blah*, **if** the object *so-and-so* possesses the property *blih-blih-blih*, **then** the object *so-and-so* possesses the property *bleh-bleh-bleh*.'

Example ($\widehat{\mathbf{A}}$). The respective converses of the conditional statements listed in Example (A) read:

(a) **Let** A be an $(n \times n)$-square matrix. **Suppose** $A = \mathcal{O}_{n\times n}$. **Then** A is symmetric and A is skew-symmetric.

(b) **Let** A, B, C be $(n \times n)$-square matrices. **Suppose** $B = C$. **Then** each of B, C is a matrix inverse of A.

(c) **Let** A be an $(n \times n)$-square matrix. **Suppose** A is invertible. **Then** $A - I_n$ is idempotent.

(d) **Let** A be an $(n \times n)$-square matrix. **Suppose** there exists some non-zero vector $\mathbf{v}$ in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$. **Then** A is idempotent, and A is not the identity matrix.

(e) **Let** A be an $(n \times n)$-square matrix. **Suppose** $I_n - A$ is invertible, and there is some positive integer k so that $I_n + A + A^2 + \cdots + A^k$ is a matrix inverse of $I_n - A$. **Then** A is not the zero matrix and A is nilpotent.

(f) **Let** A, B be $(n \times n)$-square matrices. **Suppose** for any positive integer p, $A^k B = BA^k$. **Then** $[A, B] = \mathcal{O}_{n\times n}$.

(g) **Let** A be an $(n \times n)$-square matrix. **Suppose** A is not invertible. **Then** A is nilpotent.

(h) **Let** A be an $(n \times n)$-square matrix. **Suppose** A is not nilpotent. **Then** A is idempotent and A is not the zero matrix.

Remark.
In general, a conditional statement and its converse have no relations. They are two distinct statements, with distinct mathematical content. Any one of the following scenario can take place:

- Both the conditional statement and its converse are true.
  Example. Both Q and $\widehat{Q}$ are true:
  (Q) Let A, B be $(n\times n)$-square matrices. Suppose $[A, B] = \mathcal{O}_{n\times n}$. Then for any positive integer p, $A^k B = BA^k$.
  ($\widehat{Q}$) Let A, B be $(n\times n)$-square matrices. Suppose for any positive integer p, $A^k B = BA^k$. Then $[A, B] = \mathcal{O}_{n\times n}$.
- Both the conditional statement and its converse are false.
  Example. Both R and $\widehat{R}$ are false:
  (R) Let A be an $(n \times n)$-square matrix. Suppose $A - I_n$ is idempotent. Then A is invertible.
  ($\widehat{R}$) Let A be an $(n \times n)$-square matrix. Suppose A is invertible. Then $A - I_n$ is idempotent.
- Of the conditional statement and its converse, one is true and the other is false.
  Example. S is true and $\widehat{S}$ is false:
  (S) Let A be an $(n \times n)$-square matrix. Suppose A is symmetric. Then A is idempotent.
  ($\widehat{S}$) Let A be an $(n \times n)$-square matrix. uppose A is idempotent. Then A is symmetric.

7. **Logical equivalence.**
   Consider the conditional statement

   ($\star$) '**Let** the object *so-and-so* be amongst the objects *blah-blah-blah*. **Suppose** the object *so-and-so* possesses the property *bleh-bleh-bleh*. **Then** the object *so-and-so* possesses the property *blih-blih-blih*.'

   and its converse

   ($\widehat{\star}$) '**Let** the object *so-and-so* be amongst the objects *blah-blah-blah*. **Suppose** the object *so-and-so* possesses the property *blih-blih-blih*. **Then** the object *so-and-so* possesses the property *bleh-bleh-bleh*.'

   In the scenario in which both ($\star$) and ($\widehat{\star}$) are true, stating

   ($\sharp$) 'The object *so-and-so* (from amongst the objects *blah-blah-blah*) possesses the property *bleh-bleh-bleh*.'

   will be the same as stating

   ($\flat$) 'The (same) object *so-and-so* (from amongst the objects *blah-blah-blah*) possesses the property *blih-blih-blih*.'

   We shall say ($\sharp$) and ($\flat$) are logically equivalent, and we may to present this 'logical equivalence' by combining ($\star$) and ($\widehat{\star}$) into the statement

   ($\star\star$) '**Let** the object *so-and-so* be amongst the objects *blah-blah-blah*. The object *so-and-so* possesses the property *bleh-bleh-bleh* **if and only if** the object *so-and-so* possesses the property *blih-blih-blih*.'

   We may also present ($\star\star$) as:

   ($\star\star'$) '**For any** object *so-and-so* amongst the objects *blah-blah-blah*, the object *so-and-so* possesses the property *bleh-bleh-bleh* **if and only if** the object *so-and-so* possesses the property *blih-blih-blih*.'

   Or as:

   ($\star\star''$) '**Let** the object *so-and-so* be amongst the objects *blah-blah-blah*. The statements ($\sharp$), ($\flat$) are **logically equivalent**:
   ($\sharp$) The object *so-and-so* possesses the property *bleh-bleh-bleh*
   ($\flat$) The object *so-and-so* possesses the property *blih-blih-blih*.'

   Example (AA).

   (a) It happens that both of the conditional statements are true:
   (T) '**Let** A be an $(n\times n)$-square matrix. **Suppose** A is symmetric and A is skew-symmetric. **Then** $A = \mathcal{O}_{n\times n}$.'
   ($\widehat{T}$) '**Let** A be an $(n\times n)$-square matrix. **Suppose** $A = \mathcal{O}_{n\times n}$. **Then** A is symmetric and A is skew-symmetric.'
   For this reason, we may combine T and $\widehat{T}$ into the statement
   (TT) '**Let** A be an $(n \times n)$-square matrix.
   A is symmetric and A is skew-symmetric **if and only if** $A = \mathcal{O}_{n\times n}$.'

   (b) It happens that both of the conditional statements are true:
   (U) '**Let** A, B be $(n \times n)$-square matrices. **Suppose** $[A, B] = \mathcal{O}_{n\times n}$. **Then** for any positive integer p, $A^k B = BA^k$.'
   ($\widehat{U}$) '**Let** A, B be $(n \times n)$-square matrices. **Suppose** for any positive integer p, $A^k B = BA^k$. **Then** $[A, B] = \mathcal{O}_{n\times n}$.'
   For this reason, we may combine U and $\widehat{U}$ into the statement
   (UU) '**Let** A, B be $(n \times n)$-square matrices.
   $[A, B] = \mathcal{O}_{n\times n}$ **if and only if** for any positive integer p, $A^k B = BA^k$.'