MATH1030 Row operations and matrix multiplication

1. **Definition. ('Standard base' for a 'vector space of matrices'.)**

   For each positive integer p, q, and for each $i = 1, \cdots, p$, $j = 1, \cdots, q$, we define the $(p \times q)$-matrix $E_{i,j}^{p,q}$ to be the $(p \times q)$-matrix whose (i, j)-th entry is 1 and whose other entries are all 0.

   There are altogether pq matrices $E_{i,j}^{p,q}$ as i, j vary. They are collectively referred to as the 'standard base' for the vector space of $(p \times q)$-matrices.

2. **Examples. ('Standard base' for various 'vector spaces of matrices'.)**

   (a) $E_{1,1}^{2,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{1,2}^{2,3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{1,3}^{2,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

   $E_{2,1}^{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $E_{2,2}^{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $E_{2,3}^{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (b) $E_{1,1}^{3,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{1,2}^{3,3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{1,3}^{3,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

   $E_{2,1}^{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{2,2}^{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{2,3}^{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

   $E_{3,1}^{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $E_{3,2}^{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $E_{3,3}^{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3. **Lemma (1).**

   Let p, q be positive integers. Suppose s, t are integers between 1 and p.

   Let A be a $(p \times q)$-matrix, whose (i, j)-th entry is denoted by a_{ij}.

   Then $E_{s,t}^{p,p} A$ is the $(p \times q)$-matrix whose s-th row is $\begin{bmatrix} a_{t1} & a_{t2} & \cdots & a_{tq} \end{bmatrix}$, and whose every other entry is 0.

   Remark. In plain words, multiplying $E_{s,t}^{p,p}$ to A from the left results in simultaneously 'putting' the t-th row of A into its s-th row and setting to 'zero' all other rows of A.

   Proof. For convenience, denote the (g, h)-th entry of $E_{s,t}^{p,p}$ by ε_{gh}

   For each $k = 1, 2, \cdots, q$, the (s, k)-th entry of $E_{s,t}^{p,p} A$ is the product of the s-th row of $E_{s,t}^{p,p}$ and the k-th column of A, and therefore is given by

   $$\varepsilon_{s1} a_{1k} + \varepsilon_{s2} a_{2k} + \cdots + \varepsilon_{sp} a_{pk} = a_{tk}.$$

   Hence the s-th row of $E_{s,t}^{p,p} A$ is $\begin{bmatrix} a_{t1} & a_{t2} & \cdots & a_{tq} \end{bmatrix}$.

   Whenever $g \neq s$, we have $\varepsilon_{gh} = 0$ for each h. Then, no matter which k is, the (g, k)-th entry of $E_{s,t}^{p,p} A$ is a sum of p copies of 0's, and hence is 0.

4. **Examples. (Illustrations of Lemma (1).)**

   (a) Suppose A is the (3×4)-matrix whose (i, j)-th entry is given by a_{ij}. Then:

   i. $E_{1,2}^{3,3} A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   ii. $E_{3,1}^{3,3} A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$.

   iii. $E_{3,1}^{3,3} A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

   (b) Suppose A is the (4×6)-matrix whose (i, j)-th entry is given by a_{ij}. Then:

   i. $E_{2,4}^{4,4} A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   ii. $E_{3,2}^{4,4} A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   iii. $E_{4,1}^{4,4} A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$.

5. **Lemma (2).**

   Let A be an (p, q)-matrix. Let i, k be integers between 1 and p.

   (a) *For any real number α, the resultant of the row operation $\alpha R_i + R_k$ on A is $(I_p + \alpha E^{p,p}_{k,i})A$.*

   (b) *For any non-zero real number β, the resultant of the row operation βR_k on A is $(I_p + (\beta - 1)E^{p,p}_{k,k})A$.*

   (c) *The resultant of the row operation $R_i \leftrightarrow R_k$ on A is $(I_p - E^{p,p}_{i,i} - E^{p,p}_{k,k} + E^{p,p}_{i,k} + E^{p,p}_{k,i})A$.*

   Proof. Exercise. (Straightforward calculation with the help of Lemma (1).)

6. **Examples. (Illustrations of Lemma (2).)**

   Suppose A is the (3×4)-matrix whose (i, j)-th entry is given by a_{ij}. Then:

   (a)

$$
\begin{aligned}
A \xrightarrow{4R_2+R_1} & \begin{bmatrix} 4a_{21}+a_{11} & 4a_{22}+a_{12} & 4a_{23}+a_{13} & 4a_{24}+a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\
= & \; 4\begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = 4E^{3,3}_{1,2}A + A = (I_3 + 4E^{3,3}_{1,2})A
\end{aligned}
$$

   (b)

$$
\begin{aligned}
A \xrightarrow{R_1 \leftrightarrow R_3} & \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \\
= & \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\
& + \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \\
= & \; A - E^{3,3}_{1,1}A - E^{3,3}_{3,3}A + E^{1,3}_{1,1}A + E^{3,1}_{3,3}A = (I_3 - E^{3,3}_{1,1} - E^{3,3}_{3,3} + E^{3,3}_{1,3} + E^{3,1}_{3,3})A.
\end{aligned}
$$

   (c)

$$
\begin{aligned}
A \xrightarrow{5R_2} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 5a_{21} & 5a_{22} & 5a_{23} & 5a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\
= & \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} + 4\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} = A + 4E^{3,3}_{2,2}A = (I_3 + 4E^{3,3}_{2,2})A
\end{aligned}
$$

7. **Definition. (Row operation matrices.)**

   Let p be a positive integer, and M be a (p, p)-square matrix.

   The matrix M is called a row-operation matrix of size p if any one of the statements below holds:

   (a) *$M = I_p + \alpha E^{p,p}_{k,i}$ for some real number α and some integers i, k between 1 and p.*

   (b) *$M = I_p + (\beta - 1)E^{p,p}_{k,k}$ for some non-zero real number β and some integer k between 1 and p.*

   (c) *$M = I_p - E^{p,p}_{i,i} - E^{p,p}_{k,k} + E^{p,p}_{i,k} + E^{p,p}_{k,i}$ for some integers i, k.*

   Remark. Now we know that the effect of applying a certain row operation on a matrix, say, A, is the same as multiplying A by some row operation matrix from the left.

   In fact such a square matrix is uniquely determined by the row operation concerned; it is independent of A.

   Theorem (3) below describes a 'dictionary' between the collection of all row operations on matrices with p rows and the collection of all row-operation matrices of size p. This 'dictionary' tells us the 'application of row operations' and the 'multiplication from the left by row-operation matrices' are two ways of thinking about the same thing.

8. **Theorem (3). ('Dictionary' between row operations and matrix multiplication from the left.)**

   Let p, q be positive integers.

   For any row operation ρ on $(p \times q)$-matrices, there exists some unique $(p \times p)$-square matrix $M[\rho]$ such that for any $(p \times q)$-matrix A, the matrix $M[\rho]A$ is the resultant of the application of ρ on A.

Proof. A tedious word game, with reference to the definitions for the notion of row operations and for the notion of row-operation matrices. (For MATH/BMED students, this is an exercise in MATH1050. First find out what is required to be proved.)

Remark. The table below summarizes the correspondence between row operations and row-operation matrices:

Row operation changing C to C'.	How C' is obtained from C through row-operation matrix.	'Reverse' row operation changing C' to C.	How C is recovered from C' through row-operation matrix.
$C \xrightarrow{\alpha R_i + R_k} C'$.	$C' = (I_p + \alpha E_{k,i}^{p,p})C$	$C' \xrightarrow{-\alpha R_i + R_k} C$.	$C = (I_p - \alpha E_{k,i}^{p,p})C'$
$C \xrightarrow{\beta R_k} C'$.	$C' = [I_p + (\beta - 1)E_{k,k}^{p,p}]C$	$C' \xrightarrow{(1/\beta)R_k} C$.	$C = [I_p + (1/\beta - 1)E_{k,k}^{p,p}]C'$
$C \xrightarrow{R_i \leftrightarrow R_k} C'$.	$C' = (I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p})C$	$C' \xrightarrow{R_i \leftrightarrow R_k} C$.	$C = (I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p})C'$

9. **Corollary (4).**

Let $C_1, C_2, \cdots, C_N$ be $(p \times q)$-matrices.

Suppose C_1 is row-equivalent to C_N, and are joint by some sequence of row operations $\rho_1, \rho_2, \cdots, \rho_{N-1}$:

$$C_1 \underset{\rho_1}{\longrightarrow} C_2 \underset{\rho_2}{\longrightarrow} \cdots \underset{\rho_{N-2}}{\longrightarrow} C_{N-1} \underset{\rho_{N-1}}{\longrightarrow} C_N$$

Then there exist row-operation matrices $H_1, H_2, \cdots, H_{N-1}$ of size p such that $C_N = H_{N-1}H_{N-2}\cdots H_2H_1C_1$.

Proof. This is an immediate consequence of Theorem (3).

10. **Examples. (Illustrations on Corollary (4).)**

(a) The sequence of row operations below joins C and C'':

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_2+R_1} C'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Then $C'' = H_2H_1C$, in which $H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

So $C'' = HC$, in which $H = H_2H_1 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) The sequence of row operations below joins C and C'':

$$C = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_1} C'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Then $C'' = H_2H_1C$, in which $H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $H_2 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

So $C'' = HC$, in which $H = H_2H_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) The sequence of row operations below joins C and C'':

$$C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}.$$

Then $C'' = H_2H_1C$, in which $H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

So $C'' = HC$, in which $H = H_2H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

(d) The sequence of row operations below joins C and C''':

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_3} C'' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} C''' = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Then $C''' = H_3H_2H_1C$, in which $H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $H_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

So $C''' = HC$, in which $H = H_3H_2H_1 = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.