MATH1030 Classification of systems of linear equations in terms of reduced row-echelon forms

1. **Question.**

   Why are we interested in row-echelon forms and reduced row-echelon forms? What is the relevance of Gaussian elimination for matrices to systems of linear equations?

   Answer.

   Suppose (S) is a system of linear equations and C is the augmented matrix representation of (S).

   Further suppose Gaussian elimination is applied to C to give the sequence of row operations

   $$C = C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_{N-1} \longrightarrow C_N = C'$$

   which ends at the reduced row-echelon form C' which is row-equivalent to C, and $(S_2), \cdots, (S_N)$ are the systems of linear equations whose respective augmented matrix representation are $C_2, \cdots, C_N$.

   Then the systems $(S_2), \cdots, (S_N)$ are what we will obtain in succession, with one 'equation operation' each time, starting from the (S), towards some system, namely (S_N), from which we can read off all solutions (if any at all) of S easily.

   Remark. We have established a dictionary which helps us translate between 'solving systems of linear equations' and 'finding reduced row-echelon forms for matrices'.

2. **Theorem (1). (Re-formulations of consistency for systems of linear equations.)**

   Suppose (S) is a system of m linear equations with n unknowns, and C is the augmented matrix representation of the system (S).

   Suppose C' is the reduced row-echelon form which is row-equivalent to C.

   Then the statements below are logically equivalent:

   (a) *(S) is consistent.*
   (b) *The last column of C' is not a pivot column.*
   (c) *No row of C' reads as $\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$.*

   Remark. By logic, these three statements below are logically equivalent:

   (a) (S) is inconsistent.
   (b) The last column of C' is a pivot column.
   (c) Some row of C' reads as $\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$.

3. **Illustrations for Theorem (1).**

   (a) (Refer to Example (2) in the handout *What is solving a system of linear equations*, and to Example ($\sharp$) in the handout *Row-echelon forms and reduced row-echelon forms.*)

   Consider the system (S) : $\left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array} \right.$. The reduced row-echelon form C' which is row equivalent to the augmented matrix representation C of the system (S) is given by

   $$C' = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

   The system (S) is consistent.

   (b) (Refer to Example (3) in the handout *What is solving a system of linear equations*, and to Example ($\sharp$) in the handout *Row-echelon forms and reduced row-echelon forms.*)

   Consider the system (S) : $\left\{ \begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 7 \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 \end{array} \right.$. The reduced row-echelon form C' which is row equivalent to the augmented matrix representation C of the system (S) is given by

   $$C' = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   The system (S) is inconsistent.

(c) (Refer to Example (4) in the handout *What is solving a system of linear equations*, and to Example ($\sharp$) in the handout *Row-echelon forms and reduced row-echelon forms.*)

Consider the system (S) : $\left\{\begin{array}{rcrcrcr} & & x_2 & - & 2x_3 & = & 1 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 \end{array}\right.$. The reduced row-echelon form C' which is row equivalent to the augmented matrix representation C of the system (S) is given by

$$C' = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system (S) is consistent.

4. **Theorem (2). (Classification of consistent systems of linear equations.)**

Suppose (S) is a system of m linear equations with n unknowns, and C is the augmented matrix representation of (S).

Suppose C' is the reduced row-echelon form which is row-equivalent to C. Denote the rank of C' by r.

Suppose (S) is consistent.

Then $r \leq n$, and the pivot columns of C' are within the first n columns of C' from the left. Furthermore:—

(a) *If $r = n$ then C' reads as*

$$C' = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & u_1 \\ 0 & 1 & \cdots & 0 & u_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & u_n \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{array}\right]$$

and $(x_1, x_2, \cdots, x_n) = (u_1, u_2, \cdots, u_n)$ is the unique solution of (S).

(b) *Suppose $r < n$. Then exactly $n - r$ columns of C' amongst the first n columns of C' from the left are free columns, and C' reads as*

$$C' = \left[\begin{array}{c|ccc|c|ccc|c|ccc|c|ccc|c} 1 & \star & \cdots & \star & 0 & \star & \cdots & \star & 0 & \cdots & \cdots & \cdots & 0 & \star & \cdots & \star & u_1 \\ \hline 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & 0 & \cdots & \cdots & \cdots & 0 & \star & \cdots & \star & u_2 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & \cdots & \cdots & 0 & \star & \cdots & \star & u_3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \ddots & & & & & & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & & & \ddots & & & & & \vdots \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & u_r \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

There are infinitely many solutions for (S). One of these solutions is given by '$x_{d_1} = u_1$, $x_{d_2} = u_2$, ..., $x_{d_r} = u_r$ and $x_{f_1} = \cdots = x_{f_{n-r}} = 0$'.

There are some numbers $\sharp_{k,\ell}$, for which k runs from 1 to r and ℓ runs from 1 to $n - r$, so that the solutions of (S) are described by

$$\left\{\begin{array}{lcl} x_{d_1} & = & u_1 + \sharp_{1,1}t_1 + \sharp_{1,2}t_2 + \cdots + \sharp_{1,n-r}t_{n-r} \\ x_{d_2} & = & u_2 + \sharp_{2,1}t_1 + \sharp_{2,2}t_2 + \cdots + \sharp_{2,n-r}t_{n-r} \\ & \vdots & \\ x_{d_r} & = & u_r + \sharp_{r,1}t_1 + \sharp_{r,2}t_2 + \cdots + \sharp_{r,n-r}t_{n-r} \\ x_{f_1} & = & t_1 \\ x_{f_2} & = & t_2 \\ & \vdots & \\ x_{f_{n-r}} & = & t_{n-r} \end{array}\right.$$

where $t_1, t_2, \cdots, t_{n-r}$ are arbitrary numbers.

Remark. In the context of Theorem (1), we will refer to the x_{f_i}'s as the free variables of the system, and the x_{d_j}'s as dependent variables of the system.

5. **Illustrations for Theorem (2).**

(a) (Refer to Example (1) in the handout *What is solving a system of linear equations.*)

Consider the system (S) : $\left\{\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 3 \\ 2x_1 & - & x_2 & = & 4 \end{array}\right.$ The reduced row-echelon form C' which is row equivalent to the augmented matrix representation C of the system (S) is given by

$$C' = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right].$$

The system (S) is consistent, and the rank of C' is the same as the number of unknowns in the system (S).

The system (S) has a unique solution, namely $(x_1, x_2) = (3, -1)$.

(b) (Refer to Example (2) in the handout *What is solving a system of linear equations*, and to Example ($\sharp$) in the handout *Row-echelon forms and reduced row-echelon forms.*)

Consider the system (S) : $\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array}\right.$. The reduced row-echelon form C' which is row equivalent to the augmented matrix representation C of the system (S) is given by

$$C' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

The system (S) is consistent, and the rank of C' is the same as the number of unknowns in the system (S).

The system (S) has a unique solution, namely $(x_1, x_2, x_3) = (2, -3, 4)$.

(c) (Refer to Example (4) in the handout *What is solving a system of linear equations*, and to Example ($\sharp$) in the handout *Row-echelon forms and reduced row-echelon forms.*)

Consider the system (S) : $\left\{\begin{array}{rcrcrcr} & & x_2 & - & 2x_3 & = & 1 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 \end{array}\right.$. The reduced row-echelon form C' which is row equivalent to the augmented matrix representation C of the system (S) is given by

$$C' = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system (S) is consistent, and the rank of C' is strictly less than the number of unknowns in the system (S).

The system (S) has infinitely many solutions, described by $(x_1, x_2, x_3) = (2-t, 1+2t, t)$ where t is an arbitrary real number.

(d) (Refer to Example (5) in the handout *What is solving a system of linear equations*, and to Example ($\sharp$) in the handout *Row-echelon forms and reduced row-echelon forms.*)

Consider the system (S) : $\left\{\begin{array}{rcrcrcrcrcr} & & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & + & 3x_5 & = & 4 \\ -2x_1 & - & x_2 & - & 3x_3 & + & 3x_4 & + & x_5 & = & 3 \end{array}\right.$. The reduced row-echelon form C' which is row equivalent to the augmented matrix representation C of the system (S) is given by

$$C' = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right].$$

The system (S) is consistent, and the rank of C' is strictly less than the number of unknowns in the system (S).

The system (S) has infinitely many solutions, described by $(x_1, x_2, x_3, x_4, x_5) = (10-s-t, -8-s, s, 5-t, t)$where s, t are arbitrary numbers.

6. **Corollary (3). (A special case of Theorem (2).)**

Suppose (S) is a system of m linear equations with n unknowns, and C is the augmented matrix representation of (S).

Suppose C' is the reduced row-echelon form which is row-equivalent to C. Denote the rank of C' by r.

Suppose (S) is consistent.

Suppose $m \leq n$. (So there are at most the same number of equations as there are unknowns.)

Then $r \leq m \leq n$. Furthermore:—

(a) *If $r = m = n$ then (S) has a unique solution.*

(b) *If $r < m$ or $m < n$ then (S) has infinitely many solutions.*

Remark. Illustrations for Theorem (2) also serve as illustrations for this corollary. For each of the system concerned, the number of equations in the system is at most that of the number of unknowns.