MMAT5390: Mathematical Image Processing

Midterm solutions

1. (a) Note that H is a 9×9 matrix; hence it represents a linear transformation on 3×3 images.
Obviously, H is block-circulant. Denote $H=(A_{ij})$, then
$A_{11}=A_{22}=A_{33}=\begin{pmatrix}2&3&1\\1&2&3\\3&1&2\end{pmatrix}$, $A_{12}=A_{23}=A_{31}=\begin{pmatrix}6&9&3\\3&6&9\\9&3&6\end{pmatrix}$
and $A_{13}=A_{21}=A_{32}=\begin{pmatrix}4&6&2\\2&4&6\\6&2&4\end{pmatrix}$. And these three matrices are all circulant. Hence h is shift-invariant with h_s being 3-periodic in both arguments.
But H is not a kronecker product of two 3×3 matrices; explicitly,

$$H=\begin{pmatrix}1&3&2\\2&1&3\\3&2&1\end{pmatrix}\otimes\begin{pmatrix}2&3&1\\1&2&3\\3&1&2\end{pmatrix}.$$

Hence h is separable.

(b) Note that H is a 9×9 matrix; hence it represents a linear transformation on 3×3 images.
H is not block-circulant. For example, consider the $y=1,\beta=1$-submatrix of H, i.e. $\begin{pmatrix}e^{-1}&e^{-2}&e^{-3}\\e^{-4}&e^{-5}&e^{-6}\\e^{-7}&e^{-8}&e^{-9}\end{pmatrix}$, which is not a circulant matrix, as the shift-operator t maps $\begin{pmatrix}e^{-1}\\e^{-4}\\e^{-7}\end{pmatrix}$ to $\begin{pmatrix}e^{-7}\\e^{-1}\\e^{-4}\end{pmatrix}$ instead of $\begin{pmatrix}e^{-2}\\e^{-5}\\e^{-8}\end{pmatrix}$. Hence h is not shift-invariant with h_s being 3-periodic in both arguments.
H is the kronecker product of two 3×3 matrices; explicitly,

$$H=\begin{pmatrix}1&0&e^{-1}\\e^{-1}&1&0\\0&e^{-1}&1\end{pmatrix}\otimes\begin{pmatrix}e^{-1}&e^{-2}&e^{-3}\\e^{-4}&e^{-5}&e^{-6}\\e^{-7}&e^{-8}&e^{-9}\end{pmatrix}.$$

Hence h is separable.

(c) Note that H is a 9×9 matrix; hence it represents a linear transformation on 3×3 images.
H is not block-circulant. Denote $H=(A_{ij})$. Although each A_{ij} itself is a circulant matrix, H is not block-circulant since $A_{12}\neq A_{23}\neq A_{31}$ and $A_{21}\neq A_{32}\neq A_{13}$. Hence h is not shift-invariant with h_s being 3-periodic in both arguments.
H is the kronecker product of two 3×3 matrices; explicitly,

$$H=\begin{pmatrix}0&1&0\\\sqrt{2}&0&\sqrt{3}\\0&\sqrt{5}&0\end{pmatrix}\otimes\begin{pmatrix}\sqrt{2}&0&\sqrt{3}\\\sqrt{3}&\sqrt{2}&0\\0&\sqrt{3}&\sqrt{2}\end{pmatrix}.$$

Hence h is separable.

(d) Note that H is a 4×4 matrix; hence it represents a linear transformation on 2×2 images.
Denote $H=(A_{ij})$. Obviously H is block-circulant. The $y=1,\beta=1$- and the $y=2,\beta=2$-submatrices of H are both $A_{11}=A_{22}=\begin{pmatrix}\cos(x)&\cos(2x)\\\cos(2x)&\cos(x)\end{pmatrix}$, which is circulant;

the $y=2,\beta=1$- and the $y=1,\beta=2$-submatrices of h are both $A_{12}=A_{21}=\begin{pmatrix}\sin(x) & \sin(2x)\\ \sin(2x) & \sin(x)\end{pmatrix}$, which is also circulant. Hence h is shift-invariant with h_s being 2-periodic in both arguments.

H is not a kronecker product of two 2×2 matrices. for example, consider the $y=1,\beta=1$- and $y=1,\beta=2$-submatrices of H, i.e. $\begin{pmatrix}\cos(x) & \cos(2x)\\ \cos(2x) & \cos(x)\end{pmatrix}$ and $\begin{pmatrix}\sin(x) & \sin(2x)\\ \sin(2x) & \sin(x)\end{pmatrix}$. neither is a scalar multiple of the other. Hence h is not separable.

(e) Note that H is a 4×4 matrix; hence it represents a linear transformation on 2×2 images.

H is not block-circulant. For example, consider the $y=1,\beta=1$-submatrix of H, i.e. $\begin{pmatrix}a & 0\\ b & c\end{pmatrix}$, which is not a circulant matrix, as the shift-operator t maps $\begin{pmatrix}a\\ b\end{pmatrix}$ to $\begin{pmatrix}b\\ a\end{pmatrix}$ instead of $\begin{pmatrix}0\\ c\end{pmatrix}$, which is different from $\begin{pmatrix}b\\ a\end{pmatrix}$ since $b\neq 0$.

Hence h is not shift-invariant with h_s being 2-periodic in both arguments.

H is not a kronecker product of two 2×2 matrices. for example, consider the $y=1,\beta=1$- and $y=2,\beta=2$-submatrices of h, i.e. $\begin{pmatrix}a & 0\\ b & c\end{pmatrix}$ and $\begin{pmatrix}d & e\\ 0 & f\end{pmatrix}$. neither is a scalar multiple of the other since $a,b,c,d,e,f\in\mathbb{N}\backslash\{0\}$. hence h is not separable.

2. (a) Take 13 October, 1990 for an example. Then $abcdefgh=19901013$ and $I=\begin{pmatrix}1 & 1 & 9 & 9\\ 9 & 9 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 3 & 3\end{pmatrix}$

(b) Recall that the Walsh functions are defined recursively as follows:

$$W_{2j+q}(t)\equiv(-1)^{\lfloor\frac{j}{2}\rfloor+q}\{W_j(2t)+(-1)^{j+q}W_j(2t-1)\},$$

where $\lfloor\frac{j}{2}\rfloor$ = largest integer smaller or equal to $\frac{j}{2}$; $q=0$ or 1; $j=0,1,2,\cdots$ and

$$W_0(t)\equiv\begin{cases}1 & \text{if } 0\leq t<1\\ 0 & \text{elsewhere}\end{cases}.$$

Then,

$$\begin{aligned}W_1(t)=W_{2\times0+1}(t)&=(-1)^1\{W_0(2t)+(-1)^1W_0(2t-1)\}\\ &=\begin{cases}-1 & \text{if } 0\leq t<1/2\\ 1 & \text{if } 1/2\leq t<1\\ 0 & \text{elsewhere}\end{cases}\end{aligned}$$

$$\begin{aligned}W_2(t)=W_{2\times1+0}(t)&=(-1)^0\{W_1(2t)+(-1)^1W_1(2t-1)\}\\ &=\begin{cases}-1 & \text{if } 0\leq t<1/4\\ 1 & \text{if } 1/4\leq t<3/4\\ -1 & \text{if } 3/4\leq t<1\\ 0 & \text{elsewhere}\end{cases}\end{aligned}$$

$$\begin{aligned}W_3(t)=W_{2\times1+1}(t)&=(-1)^1\{W_1(2t)+(-1)^2W_1(2t-1)\}\\ &=\begin{cases}1 & \text{if } 0\leq t<1/4\\ -1 & \text{if } 1/4\leq t<1/2\\ 1 & \text{if } 1/2\leq t<3/4\\ -1 & \text{if } 3/4\leq t<1\\ 0 & \text{elsewhere}\end{cases}\end{aligned}$$

Since $W = \frac{1}{\sqrt{N}}(w_{ki}) = \frac{1}{\sqrt{N}}(W_k(\frac{i}{N}))$ where $k, i = 0, 1, ..., N-1$.

$$W = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

(c)

$$W^TW = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = I_4$$

Therefore, W is orthogonal.

(d)

$$\begin{aligned} I_{Walsh} &= WIW^T \\ &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 9 & 9 \\ 9 & 9 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ -2 & -10 & 0 & 0 \\ -1 & 7 & 0 & 0 \end{pmatrix} \end{aligned}$$

3. (a) i. $A^TA = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 25 \end{pmatrix}$

Hence the characteristic polynomial of A^TA is given by

$\det(A^TA - \lambda I_3) = \begin{vmatrix} 10-\lambda & 6 & 0 \\ 0 & 10-\lambda & 0 \\ 0 & 0 & 25-\lambda \end{vmatrix} = -(\lambda-25)(\lambda-16)(\lambda-4).$

Therefore the eighenvalues of A^TA are $25, 16, 4$.

ii. From the first part we know that the eigenvalues of A^TA are:

$$\lambda_1 = 25, \quad \lambda_2 = 16, \quad \lambda_3 = 4$$

For $\lambda_1 = 25$,

$$[A^TA - \lambda_1 I_3|0] = \left[\begin{array}{ccc|c} -15 & 6 & 0 & 0 \\ 6 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector, which is also unit.

For $\lambda_2 = 16$,

$$[A^TA - \lambda_2 I_3|0] = \left[\begin{array}{ccc|c} -6 & 6 & 0 & 0 \\ 6 & -6 & 0 & 0 \\ 0 & 0 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector, which gives the unit eigenvector $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = 4$,

$$[A^TA - \lambda_3 I_3|0] = \left[\begin{array}{ccc|c} 6 & 6 & 0 & 0 \\ 6 & 6 & 0 & 0 \\ 0 & 0 & 21 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ is an eigenvector, which gives the unit eigenvector $\vec{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$.

Then $\vec{u}_1 = \frac{1}{\sqrt{\lambda_1}}A\vec{v}_1 = \frac{1}{\sqrt{25}}\begin{pmatrix}3&1&0\\1&3&0\\0&0&5\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$,

$\vec{u}_2 = \frac{1}{\sqrt{\lambda_2}}A\vec{v}_2 = \frac{1}{\sqrt{16}}\begin{pmatrix}3&1&0\\1&3&0\\0&0&5\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$,

$\vec{u}_3 = \frac{1}{\sqrt{\lambda_3}}A\vec{v}_3 = \frac{1}{\sqrt{4}}\begin{pmatrix}3&1&0\\1&3&0\\0&0&5\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$,

Hence an svd of A is given by $A = U\Sigma V^T$, where $U = (\vec{u}_1, \vec{u}_2, \vec{u}_3) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&1\\0&1&-1\\\sqrt{2}&0&0\end{pmatrix}$,

$\Sigma = \begin{pmatrix}\sqrt{25}&0&0\\0&\sqrt{16}&0\\0&0&\sqrt{4}\end{pmatrix} = \begin{pmatrix}5&0&0\\0&4&0\\0&0&2\end{pmatrix}$ and $V = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&1\\0&1&-1\\\sqrt{2}&0&0\end{pmatrix}$.

iii. The elementary images according to the above svd are given by:

$$\vec{u}_1\vec{v}_1^T = \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0&0&1\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix};$$

$$\vec{u}_2\vec{v}_2^T = \frac{1}{\sqrt{4}}\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1&1&0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix};$$

$$\vec{u}_3\vec{v}_3^T = \frac{1}{\sqrt{4}}\begin{pmatrix}1\\-1\\0\end{pmatrix}\begin{pmatrix}1&-1&0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix};$$

Hence,

$$\begin{aligned}A &= \sqrt{25}\vec{u}_1\vec{v}_1^T + \sqrt{16}\vec{u}_2\vec{v}_2^T + \sqrt{4}\vec{u}_3\vec{v}_3^T\\ &= 5\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} + 4\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&0\\0&0&0\end{pmatrix} + 2\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}&0\\-\frac{1}{2}&\frac{1}{2}&0\\0&0&0\end{pmatrix}\end{aligned}$$

iv. Since $A = U\Sigma V^T$, where both U and V are orthogonal matrices, and $\Sigma = \begin{pmatrix}5&0&0\\0&4&0\\0&0&2\end{pmatrix}$.

We modified Σ to obtain $\tilde{\Sigma} = \begin{pmatrix}5&0&0\\0&4&0\\0&0&0\end{pmatrix}$, then $\tilde{A} = U\tilde{\Sigma}V^T = \begin{pmatrix}2&2&0\\2&2&0\\0&0&5\end{pmatrix}$.

Obviously, $\|\tilde{A} - A\|_F = \|\tilde{\Sigma} - \Sigma\|_F = 2$ and rank($\tilde{A}$)=rank($\tilde{\Sigma}$)=2

(b) Since there exits a typo in this part, we exclude it from the midterm exam and include it into the special assignment. Please refer to the solution of special assignment.

4. $\tilde{g}(k,l) = g(-1-k,-1-l)$ for any $0 \le k \le N-1, 0 \le l \le N-1$. Then

$$\begin{aligned}
\hat{\tilde{g}}(m,n) &= \frac{1}{N^2}\sum_{k=-N}^{-1}\sum_{l=-N}^{-1}\tilde{g}(k,l)e^{-2\pi j\frac{mk+nl}{N}} \\
&= \frac{1}{N^2}\sum_{k=-N}^{-1}\sum_{l=-N}^{-1}g(-1-k,-1-l)e^{-2\pi j\frac{mk+nl}{N}} \\
&= \frac{1}{N^2}\sum_{k'=0}^{N-1}\sum_{l'=0}^{N-1}g(k',l')e^{-2\pi j\frac{m(-1-k')+n(-1-l')}{N}} \\
&= \frac{1}{N^2}e^{2\pi j\frac{m+n}{N}}\sum_{k'=0}^{N-1}\sum_{l'=0}^{N-1}g(k',l')e^{2\pi j\frac{-mk'-nl'}{N}} \\
&= e^{2\pi j\frac{m+n}{N}}\hat{g}(-m,-n).
\end{aligned}$$

5. (a) Recall that

$$\hat{F}(k,l) = \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}F(m,n)e^{-j\frac{2\pi}{N}(mk+nl)}$$

$\hat{F}(k,l)$ is called the Fourier coefficient at (k,l).

Observe that: for $0 \le k,l \le \dfrac{N}{2}-1$

$$\begin{aligned}
\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}+l\right) &= \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}F(m,n)e^{-j\frac{2\pi}{N}(m(\frac{N}{2}+k)+n(\frac{N}{2}+l))} \\
&= \overline{\frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}F(m,n)(-1)^{m+n}e^{-j\frac{2\pi}{N}(m(-k)+n(-l))}} \\
&= \overline{\frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}F(m,n)e^{-j\frac{2\pi}{N}(m(\frac{N}{2}-k)+n(\frac{N}{2}-l))}} \\
&= \overline{\hat{F}\left(\frac{N}{2}-k,\frac{N}{2}-l\right)}
\end{aligned}$$

From above, we deduce that:

$$\begin{aligned}
F(m,n) = &\sum_{0\le k,l\le \frac{N}{2}-1}\left[\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}+l\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m+(\frac{N}{2}+l)n]}\right] \\
&+\sum_{1\le k,l\le \frac{N}{2}-1}\left[\overline{\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}+l\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m+(\frac{N}{2}-l)n]}\right] \\
&+\sum_{0\le k,l\le \frac{N}{2}-1}\left[\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}-l\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m+(\frac{N}{2}-l)n]}\right] \\
&+\sum_{1\le k,l\le \frac{N}{2}-1}\left[\overline{\hat{F}\left(\frac{N}{2}+k,\frac{N}{2}-l\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m+(\frac{N}{2}+l)n]}\right] \\
&+\sum_{0\le l\le \frac{N}{2}-1}\hat{F}\left(0,\frac{N}{2}+l\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+l)n]}+\sum_{1\le l\le \frac{N}{2}-1}\overline{\hat{F}\left(0,\frac{N}{2}+l\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-l)n]} \\
&+\sum_{0\le k\le \frac{N}{2}-1}\hat{F}\left(\frac{N}{2}+k,0\right)e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m]}+\sum_{1\le k\le \frac{N}{2}-1}\overline{\hat{F}\left(\frac{N}{2}+k,0\right)}e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m]}+\hat{F}(0,0)
\end{aligned}$$

When k and l are close to $\frac{N}{2}$, $\hat{F}\left(\underbrace{\frac{N}{2}+k}_{\approx N}, \underbrace{\frac{N}{2}+l}_{\approx N}\right)$ is associated to $\underbrace{e^{-j\frac{2\pi}{N}(k'm+l'n)}}_{\approx e^{-j\frac{2\pi}{N}(\tilde{k}m+\tilde{l}n)}}$, where k' and l' are close to N and $\tilde{k}$ and $\tilde{l}$ are close to 0.

Therefore, Fourier coefficients at bottom right corner are associated to low frequency components. Similarly, Fourier coefficients at the four corners are associated to low frequency components. On the other hand, Fourier coefficients in the middle are associated to high frequency components.

(b) Denote $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ by C. Then $A = \begin{pmatrix} C & & \\ & D & \\ & & C \end{pmatrix}$, $B = \begin{pmatrix} & & C \\ & E & \\ C & & \end{pmatrix} = \begin{pmatrix} & & C \\ & 0 & \\ C & & \end{pmatrix}$

Obviously, $C^2 = 3C$ and $C^k = 3^{k-1}C$ for $k \in \mathbb{N}\backslash\{0\}$.

Besides, from $D = VJV^{-1}$ we know that $D^k = (VJV^{-1})^k = VJ^kV^{-1}$ for $k \in \mathbb{N}\backslash\{0\}$.

And J^{N-6} is a zero matrix because J is a super-diagonal matrix with size $(N-6) \times (N-6)$. Therefore $D^{N-6} = VJ^{N-6}V^{-1}$ is a zero matrix.

Hence

$$A^{N-6} = \begin{pmatrix} C^{N-6} & & \\ & D^{N-6} & \\ & & C^{N-6} \end{pmatrix} = \begin{pmatrix} 3^{N-7}C & & \\ & 0 & \\ & & 3^{N-7}C \end{pmatrix}$$

And

$$\begin{aligned} H &= \frac{1}{3^{N-7}}A + B \\ &= \frac{1}{3^{N-7}}\begin{pmatrix} 3^{N-7}C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3^{N-7}C \end{pmatrix} + \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C \end{pmatrix} + \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} C & 0 & C \\ 0 & 0 & 0 \\ C & 0 & C \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

Since the four corners of $\hat{I}$ can be regarded as the low frequency region, H is exactly a low-pass filtering.