MMAT5390: Mathematical Image Processing

Assignment 4 Solutions

1. (a) Here we discretize the energy functional $E(f)$ (with forward difference for first derivatives);

$$E_{discrete}(f) = \sum_{(x,y)\in\Omega} \{|f(x,y)-g(x,y)|+\lambda[f(x+1,y)-f(x,y)]^2+\lambda[f(x,y+1)-f(x,y)]^2\}.$$

(b) Suppose f is a minimizer of $E_{discrete}$. Then, for each position $(x,y) \in S$,

$$\begin{aligned}
0 &= \frac{\partial E_{discrete}}{\partial f(x,y)} \\
&= \frac{f(x,y)-g(x,y)}{|f(x,y)-g(x,y)|+\epsilon} + 2\lambda(f(x+1,y)-f(x,y))(-1) + 2\lambda(f(x,y+1)-f(x,y))(-1) \\
&+ 2\lambda(f(x,y)-f(x-1,y)) + 2\lambda(f(x,y)-f(x,y-1)) \\
&= \frac{f(x,y)-g(x,y)}{|f(x,y)-g(x,y)|+\epsilon} + 2\lambda[4f(x,y)-f(x+1,y)-f(x,y+1)-f(x-1,y)-f(x,y-1)]
\end{aligned}$$

(c) Now we are solving:

$$\min_f E_{discrete}(f)$$

where f is a discrete image function $(x,y) \mapsto f(x,y)$. Hence, we consider $E_{discrete}$ to be a function defined on a discrete image.
Consider a time-dependent image $f(x,y;t)$. Assume $f(x,y;t)$ solve the ODE:

$$\frac{df(x,y;t)}{dt} = -\nabla E_{discrete}(f(x,y;t)) \text{ (gradient descent equation)}$$

We show that $E_{discrete}(f(x,y;t))$ is decreasing as t increases.
In fact,

$$\frac{d}{dt}E_{discrete}(f(x,y;t)) = \nabla E_{discrete}(f(x,y;t))f'(x,y;t) = -|\nabla E_{discrete}(f(x,y;t))|^2 \le 0$$

Therefore, $E_{discrete}(f(x,y;t))$ is decreasing as t increases. We solve the gradient descent equation iteratively. Let $f^0(x,y)$ be the initial guess.
We solve the ODE in a discrete sense:

$$\frac{f^{n+1}-f^n}{\Delta t} = -\nabla E_{discrete}(f^n)$$

Δt is called the <u>time step</u>, which must be chosen carefully.
In our case, it becomes:

$$\begin{aligned}
&\frac{f^{n+1}(x,y)-f^n(x,y)}{\Delta t} \\
&= -\frac{f^n(x,y)-g(x,y)}{|f^n(x,y)-g(x,y)|+\epsilon} - 2\lambda[4f^n(x,y)-f^n(x+1,y)-f^n(x,y+1)-f^n(x-1,y)-f^n(x,y-1)]
\end{aligned}$$

(d) Suppose f is a minimizer of E. Then for any $v : \Omega \to \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\bigg|_{t=0} E(f+tv) \\
&= \int_\Omega \frac{\partial}{\partial t}\bigg|_{t=0} [(f+tv-g)^2 + \lambda\|\nabla(f+tv)\|^4]\,dx\,dy \\
&= \int_\Omega [2v(f-g) + \lambda \frac{\partial}{\partial t}\bigg|_{t=0} (\langle \nabla f + t\nabla v, \nabla f + t\nabla v\rangle)^2]\,dx\,dy \\
&= \int_\Omega [2v(f-g) + 2\lambda\|\nabla f\|^2 2\langle \nabla f, \nabla v\rangle]\,dx\,dy \\
&= 2\int_\Omega v[f-g-2\nabla\cdot(\lambda\|\nabla f\|^2\nabla f)]\,dx\,dy + 4\int_{\partial\Omega} v\lambda\|\nabla f\|^2\langle \nabla f, \vec{n}\rangle\,ds.
\end{aligned}$$

Since v is arbitrarily chosen, f satisfies:

$$\begin{cases} -2\nabla\cdot(\lambda\|\nabla f\|^2\nabla f)(x,y) + f(x,y) = g(x,y) & \text{if } (x,y)\in\Omega, \\ \lambda\|\nabla f(x,y)\|^2\langle \nabla f(x,y), \vec{n}(x,y)\rangle = 0 & \text{if } (x,y)\in\partial\Omega. \end{cases}$$

a descent direction is given by

$$v(x,y) = \begin{cases} -f(x,y) + g(x,y) + 2\nabla\cdot(\lambda\|\nabla f\|^2\nabla f)(x,y) & \text{if } (x,y)\in\Omega, \\ -\lambda\|\nabla f(x,y)\|^2\langle \nabla f(x,y), \vec{n}(x,y)\rangle & \text{if } (x,y)\in\partial\Omega. \end{cases}$$

and thus $E(f)$ can be iteratively minimized by updating f:

$$f^{n+1}(x,y) = \begin{cases} f^n(x,y) + \Delta t\{-f^n(x,y) + g(x,y) + 2\nabla\cdot(\lambda\|\nabla f^n(x,y)\|^2\nabla f^n)(x,y)\} & \text{if } (x,y)\in\Omega, \\ f^n(x,y) + \Delta t\{-\lambda\|\nabla f^n(x,y)\|^2\langle \nabla f^n(x,y), \vec{n}(x,y)\rangle\} & \text{if } (x,y)\in\partial\Omega. \end{cases}$$

2. (a)

$$E_{snake,2}(\gamma) = \int_0^{2\pi} \frac{1}{2}\|\gamma'(s)\|^2\,ds + \alpha\int_0^{2\pi}\frac{1}{2}\|\gamma''(s)\|^2\,ds + \beta\int_0^{2\pi} V(\gamma(s))\,ds,$$

Given a contour $\gamma^n(t)$ at the n-th iteration. We proceed to perturb $\gamma^{n+1}(t)$ by $\gamma^{n+1} := \gamma^n + \varepsilon\varphi$ to minimize $E_{snake,2}$. We need to find φ s.t. $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E_{snake,2}(\gamma^{n+1}) < 0$.

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E_{snake,2}(\gamma^{n+1}) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_0^{2\pi}\frac{1}{2}\|(\gamma^n)'(s)+\varepsilon\varphi'(s)\|^2\ ds\ + \alpha\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_0^{2\pi}\frac{1}{2}\|(\gamma^n)''(s)+\varepsilon\varphi''(s)\|^2\ ds \\
&\quad + \beta\,\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_0^{2\pi} V(\gamma^n(s)+\varepsilon\varphi(s))\ ds \\
&= \int_0^{2\pi}(\gamma^n)'(s)\cdot\varphi'(s)\ ds\ + \alpha\int_0^{2\pi}(\gamma^n)''(s)\cdot\varphi''(s)\ ds\ + \beta\int_0^{2\pi}\nabla V(\gamma^n(s))\cdot\varphi(s)\ ds \\
&= -\int_0^{2\pi}(\gamma^n)''(s)\cdot\varphi(s)\ ds\ + \alpha\int_0^{2\pi}(\gamma^n)^{(4)}(s)\cdot\varphi(s)\ ds\ + \beta\int_0^{2\pi}\nabla V(\gamma^n(s))\cdot\varphi(s)\ ds \\
&= \int_0^{2\pi}[-(\gamma^n)''(s) + \alpha(\gamma^n)^{(4)}(s) + \beta\nabla V(\gamma^n(s))]\cdot\varphi(s)\ ds
\end{aligned}$$

In order that $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E_{snake,2}(\gamma^{n+1}) < 0$ (decreasing), we must have:

$$\varphi(s) = (\gamma^n)''(s) - \alpha(\gamma^n)^{(4)}(s) - \beta\nabla V(\gamma^n(s))$$

Thus, we must modify γ^n by:

$$\gamma^{n+1} = \gamma^n + \varepsilon((\gamma^n)''(s) - \alpha(\gamma^n)^{(4)}(s) - \beta\nabla V(\gamma^n(s))) \qquad \text{for small } \varepsilon > 0$$

or

$$\frac{\gamma^{n+1}-\gamma^n}{\varepsilon} = \underbrace{(\gamma^n)''(s) - \alpha(\gamma^n)^{(4)}(s) - \beta\nabla V(\gamma^n(s))}_{-\nabla E(\gamma^n(s))\ (definition)}$$

In the continuous setting, we aim to obtain a time-dependent contour: $\gamma_t(s) := \gamma(s;t)$ such that: $\frac{d}{dt}\gamma_t(s) = -\nabla E(\gamma_t(s))$.

(b) Let N = number of discrete points in $[0, 2\pi]$,
$\sigma = \frac{2\pi}{N}$ = step length and $s_i = i\sigma \quad (i = 1, 2, \cdots, N)$
$u_i^k = \gamma(s_i; t^k) = \gamma(i\sigma; k\tau)$ = i-th node of the contour, where τ is time step.

Define $u^k = (u_1^k, u_2^k, \cdots, u_N^k)^T \in M_{N\times 2}(\mathbb{R})$ = discrete closed curve / contour, where $u_i^k \in \mathbb{R}^2$ for all i.
The discrete derivative can be approximated by finite difference scheme:

$$\gamma_k'(i\sigma) = \frac{u_{i+1}^k - u_i^k}{\sigma} \text{ and } \gamma_k''(i\sigma) = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{\sigma^2} \qquad i = 1, 2, \cdots, N$$

Here, we assume $\gamma_{N+1}^k = \gamma_1^k$; $\gamma_{N+2}^k = \gamma_2^k$; $\gamma_N^k = \gamma_0^k$; $\gamma_{N-1}^k = \gamma_{-1}^k$ (since the contour is closed).
Thus, the discrete snake energy can be written as:

$$E_{snake,2}(u) = \sum_{i=1}^{N} \frac{1}{2} \left\| \frac{u_{i+1}-u_i}{\sigma} \right\|^2 \sigma + \alpha \sum_{i=1}^{N} \frac{1}{2} \left\| \frac{u_{i+1} - 2u_i + u_{i-1}}{\sigma^2} \right\|^2 \sigma + \beta \sum_{i=1}^{N} V(u_i)\sigma$$

where $u = (u_1, u_2, \cdots, u_N)^T$ is a discrete closed curve ($u_i \in \mathbb{R}^2$ for all i). We can throw away σ

$$E_{snake,2}(u) = \sum_{i=1}^{N} \frac{1}{2} \left\| \frac{u_{i+1}-u_i}{\sigma} \right\|^2 + \alpha \sum_{i=1}^{N} \frac{1}{2} \left\| \frac{u_{i+1} - 2u_i + u_{i-1}}{\sigma^2} \right\|^2 + \beta \sum_{i=1}^{N} V(u_i)$$

(c) To minimize $E_{snake,2}$, we compute ∇E and find u such that $\nabla E(u) = 0$. now,

$$\begin{aligned}\frac{\partial E}{\partial u_i} =& -\left(\frac{u_{i+1}-u_i}{\sigma^2}\right) + \left(\frac{u_i - u_{i-1}}{\sigma^2}\right) \\ &+ \frac{\alpha}{2}\left[2\frac{u_i + u_{i-2} - 2u_{i-1}}{\sigma^4} + 2\frac{u_{i+2} + u_i - 2u_{i+1}}{\sigma^4} - 4\frac{u_{i+1} + u_{i-1} - 2u_i}{\sigma^4}\right] \\ &+ \beta\nabla V(u_i) \\ =& \frac{\alpha}{\sigma^4}u_{i+2} + \frac{-\sigma^2 - 4\alpha}{\sigma^4}u_{i+1} + \frac{2\sigma^2 + 6\alpha}{\sigma^4}u_i + \frac{-\sigma^2 - 4\alpha}{\sigma^4}u_{i-1} + \frac{\alpha}{\sigma^4}u_{i-2} + \beta\nabla V(u_i)\end{aligned}$$

(Recall that: $u_i = (u_{i_1}, u_{i_2})^T \in \mathbb{R}^2$. We define: $\frac{\partial E}{\partial u_i} = \left(\frac{\partial E}{\partial u_{i_1}}, \frac{\partial E}{\partial u_{i_2}}\right)^T$.
Thus, $\frac{\partial V}{\partial u_i} = \left(\frac{\partial V}{\partial u_{i_1}}, \frac{\partial V}{\partial u_{i_2}}\right)^T = \nabla V(u_i)$)
Define:

$$D = -\begin{pmatrix} \frac{2\sigma^2+6\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{\alpha}{\sigma^4} & 0 & \cdots & 0 & \frac{\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} \\ \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{2\sigma^2+6\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{\alpha}{\sigma^4} & \cdots & 0 & 0 & \frac{\alpha}{\sigma^4} \\ \frac{\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{2\sigma^2+6\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{\alpha}{\sigma^4} & 0 & 0 & \cdots & \frac{\alpha}{\sigma^4} & \frac{-\sigma^2-4\alpha}{\sigma^4} & \frac{2\sigma^2+6\alpha}{\sigma^4} \end{pmatrix}$$

Define $F(u) = (F_1(u), F_2(u), \cdots, F_N(u))^T \in M_{N\times 2}(\mathbb{R})$
where $F_i(u) = -\nabla V(u_i), \quad i = 1, 2, \cdots, N$.

Then: $\dfrac{\partial E}{\partial u_i} = -(Du)_i - \beta(F(u))_i$

Using the gradient descent method, we can minimize E_{snake} by Explicit Euler scheme:

$$\frac{u_i^{k+1} - u_i^k}{\tau} = (Du^k)_i + \beta(F(u^k))_i \qquad \tau = \text{ time step}$$

3.

$$E(I) = \int_\Omega K(x,y)\|\nabla I(x,y)\|^2 dxdy$$

For any function $\phi(x,y)$

$$\begin{aligned}
\frac{dE(I+t\phi)}{dt}\bigg|_{t=0} &= \frac{d}{dt}\bigg|_{t=0} \int_\Omega K(x,y)\|\nabla(I+t\phi)(x,y)\|^2 dxdy \\
&= \int_\Omega K(x,y)\ \frac{d}{dt}\bigg|_{t=0} \langle \nabla I + t\phi, \nabla I + t\phi\rangle(x,y)dxdy \\
&= 2\int_\Omega \langle K(x,y)\nabla I(x,y), \nabla\phi(x,y)\rangle dxdy \\
&= 2\{\int_\Omega [-\nabla\cdot(K(x,y)\nabla I(x,y)]\phi(x,y)\, dxdy + \int_{\partial\Omega} \phi(x,y)\langle K(x,y)\nabla I(x,y), \vec{n}\rangle ds\}
\end{aligned}$$

When we consider all compactly supported functions ϕ, we have

$$\frac{dE(I+t\phi)}{dt}\bigg|_{t=0} = 2\int_\Omega [-\nabla\cdot(K(x,y)\nabla I(x,y)]\phi(x,y)\, dxdy$$

Therefore, $\nabla E(I(x,y)) = -\nabla\cdot(K(x,y)\nabla I(x,y))$ for any $(x,y)\in\Omega$ So we can just consider this equation

$$\frac{\partial I(x,y;t)}{\partial t} = \nabla\cdot(K(x,y)\nabla I(x,y;t))$$

Because $\dfrac{dE(I(x,y,t))}{dt} = \nabla E(I(x,y,t)) \cdot \dfrac{d}{dt} I(x,y,t) = -|\nabla E(I(x,y,t))|^2 \leq 0.$

And $E(I(x,y,t))$ wil decrease if $I(x,y,t)$, which satisfies this equation, moves along time t