MMAT5390 Mathematical Image Processing

Final Examination

Name: ________________________ Student ID: ____________________

Please show all your steps, unless otherwise stated. Answer all **five** questions. The total score is 100. Your submitted solution will be checked carefully to avoid plagiarism. Discussions amongst classmates are strictly prohibited.

1. (**20pts**) Consider a 4×4 periodically extended image $I = (I(k,l))_{0 \le k,l \le 3}$ given by:

$$I = \begin{pmatrix} a & a-2c & 0 & 0 \\ b-2c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

where $a, b, c \ge 0$.

Consider the modified direct filter T_1 with squared radius a and order b, which is defined by

$$T_1(u,v) = \frac{B(u,v)}{H_1(u,v) + \epsilon \cdot sgn(H_1(u,v))}$$

where $B(u,v) = \dfrac{1}{1 + (\frac{u^2+v^2}{a})^b}$, $\epsilon = 1$ and $H_1(u,v) = DFT(h_1)(u,v)$ with h_1 being a bluring convolution kernel $\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Consider the constrained least square filter T_2 with parameter c, which is defined by

$$T_2(u,v) = \frac{1}{N^2} \frac{\overline{H_2(u,v)}}{|H_2(u,v)|^2 + c|P(u,v)|^2}$$

where $H_2 = DFT(h_2)$ with h_2 being the convolution kernel $\begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

and $P = DFT(\tilde{h})$ with $\tilde{h}$ being the convolution kernel $\begin{pmatrix} -4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Let $I_2(u,v) = T_2(u,v) DFT(I)(u,v)$

Suppose $DFT(I)(0,0) = \dfrac{1}{2}$; $T_1(0,2) = \dfrac{8}{17}$; $DFT(I)(1,1) \neq 0$ and $I_2(1,1) = \dfrac{3j}{73} DFT(I)(1,1)$. Find a, b, c. (Here, $j = \sqrt{-1}$.)

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2. (**20pts**) This question is about convolution and unsharp masking.

   (a) Let $D(x,y) := x^2 + y^2$.

   Let $h_1(x,y) = \begin{cases} \frac{1}{9} & \text{if } D(x,y) \leq 2, \\ 0 & \text{otherwise;} \end{cases}$ and $h_2(x,y) = \begin{cases} \frac{1}{2} & \text{if } D(x,y) = 0, \\ \frac{1}{8} & \text{if } D(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$

   Find $h \in M_{N\times N}(\mathbb{R})$, $N \geq 5$, such that $h*f = h_2*(h_1*f)$ for any $f \in M_{N\times N}(\mathbb{R})$:

   (b) Let $\tilde{h}(x,y) = \begin{cases} h(x,y) & \text{if } D(x,y) \leq 2, \\ 0 & \text{otherwise.} \end{cases}$, where h is the kernel from (a).

   Let $H(u,v)$ be the LPF such that

$$DFT(\tilde{h} * f)(u,v) = H(u,v)DFT(f)(u,v),$$

   Using H, perform unsharp masking (i.e. $k = 1$) on the following periodically extended 4×4 image.

$$f = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

3. (**15pts**) Suppose a camera undergoes a uniform planar motion during capturing a scene. More precisely, the $N \times N$ motion blurred image $I = (I(x,y))_{0\leq x,y\leq N-1}$ is given by:

$$I(x,y) = \sum_{T=0}^{L-1} f(x + mT, y + nT),$$

   where $f(x,y)$ is the static image obtained by the camera without motion, m and n are positive real numbers. Prove that:

$$DFT(I)(u,v) = H(u,v)DFT(f)(u,v), \text{ where } 0 \leq u,v \leq N-1$$

   for a degradation function $H(u,v)$. What is $H(u,v)$? Please show your answer with details.

4. (**20pts**) Given a noisy image $I : D \to \mathbb{R}$, we consider the following image denoising model to restore the original clean image $f : D \to \mathbb{R}$ that minimizes:

$$E(f) = \int_D (f(x,y) - I(x,y))^2 + \int_D e^{h(x,y)} \|\nabla f(x,y)\|^2,$$

   where $h : D \to \mathbb{R}$.

   (a) Prove that if f minimizes $E(f)$, then f satisfies the following PDE:

$$(*) \begin{cases} -\nabla \cdot \left(e^{h(x,y)} \nabla f(x,y)\right) + f(x,y) = I(x,y) & \text{for } (x,y) \in D \\ \nabla f(x,y) \cdot \vec{n}(x,y) = 0 & \text{for } (x,y) \in \partial D \end{cases}$$

   where $\vec{n}$ is the outward normal of the boundary of D.

   (b) Conversely, prove that if f satisfies (*), then f minimizes $E(f)$.

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5. (**25pts**) Consider the modified active contour model to find a parameterized closed curve $\gamma : [0, 2\pi] \to \Omega$, where $\Omega \subset \mathbb{R}^2$ is the image domain and $\gamma(0) = \gamma(2\pi)$, that minimizes:

$$E_{snake}(\gamma) = \int_0^{2\pi} \frac{1}{2}|\gamma'(s)|^2 ds + \sum_{k=2}^{N} \alpha_k \int_0^{2\pi} \frac{1}{2}|\gamma^{(k)}(s)|^2 ds + \beta \int_0^{2\pi} V(\gamma(s)) ds.$$

Here, $V : \Omega \to \mathbb{R}$ is the edge detector. α_k's and β are positive parameters and $\gamma^{(k)}(s)$ is the k-th order derivative with respect to s.

(a) Using the gradient descent method, derive an iterative scheme to minimize E_{snake} (in the continuous setting). That is, develop an iterative scheme of the form:

$$\frac{\gamma^{n+1} - \gamma^n}{\epsilon} = -\nabla E_{snake}(\gamma^n)$$

and find $-\nabla E_{snake}$. Please explains your answer with details.

(b) Suppose $\alpha_k = 0$ for $k = 2, ..., N$. And $\gamma'(s)$ can be approximated by:

$$\gamma'(s) = \lambda_1 \left(\frac{\gamma(s+h) - \gamma(s))}{h} \right) + \lambda_2 \left(\frac{\gamma(s+h) - \gamma(s-h))}{2h} \right),$$

where $\lambda_1, \lambda_2 \geq 0$. Thus, in the discrete setting, E_{snake} is discretized as:

$$\tilde{E}_{snake}(\mathbf{u}) = \sum_{i=1}^{N} \frac{1}{2} \left| \lambda_1 \left(\frac{\mathbf{u}_{i+1} - \mathbf{u}_i}{\sigma} \right) + \lambda_2 \left(\frac{\mathbf{u}_{i+1} - \mathbf{u}_{i-1}}{2\sigma} \right) \right|^2 + \beta V(\mathbf{u}_i)$$

where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N)^T \in M_{N\times 2}(\mathbb{R})$ is a discrete closed curve ($\mathbf{u}_i \in \mathbb{R}^2$ for all i) and $\sigma = \frac{2\pi}{N}$ is the arc-length parameter.

i. Derive the explicit Euler scheme to iteratively obtain a sequence of discrete curve $\{\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, ..., \mathbf{u}^{(k)}, \mathbf{u}^{(k+1)}, ...\}$ such that

$$\tilde{E}_{snake}(\mathbf{u}^{(k+1)}) \leq \tilde{E}_{snake}(\mathbf{u}^{(k)}),$$

for $k = 0, 1, 2, ...$, where $\mathbf{u}^{(k)} = (\mathbf{u}_1^{(k)}, \mathbf{u}_2^{(k)}, ..., \mathbf{u}_N^{(k)})^T \in M_{N\times 2}(\mathbb{R})$ is the discrete curve at the k-th iteration. That is, develop an iterative scheme of the form:

$$\frac{\mathbf{u}_i^{(k+1)} - \mathbf{u}_i^{(k)}}{\tau} = (G(\mathbf{u}^{(k)}))_i \text{ where } k = 0, 1, 2, ...,$$

for some suitable G and find G. Please explain your answer with details.

ii. Write down the semi-implicit scheme to obtain a sequence of discrete curves evolving to the boundary of the image object.

iii. Assume $\lambda_1 = 1$, $\lambda_2 = 0$ and $V(x, y) = x^2 + y^2$. Suppose the initial curve is given by $\mathbf{u}^{(0)} = (\mathbf{u}_1^{(0)}, \mathbf{u}_2^{(0)}, ..., \mathbf{u}_N^{(0)})^T$, where $\mathbf{u}_m^{(0)} = (\cos(\frac{2\pi m}{N}), \sin(\frac{2\pi m}{N}))^T$ for $m = 1, 2, ..., N$. Under the semi-implicit scheme, prove that $\mathbf{u}^{(k)}$ is a discrete curve representing a circle for $k = 1, 2, ...$ Show that $\mathbf{u}^{(k)}$ converges to the origin $(0, 0)$ if τ is small enough.

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