MMAT5390: Mathematical Image Processing

Assignment 3 Solutions

1. (a) For any $0 \le p, q \le N-1$,

$$\begin{aligned}
iDFT(DFT(f))(p,q) &= \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)e^{2\pi j\frac{m(k-p)+n(l-q)}{N}} \\
&= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\left[\sum_{m=0}^{N-1} e^{2\pi j\frac{m(k-p)}{N}}\right]\left[\sum_{n=0}^{N-1} e^{2\pi j\frac{n(l-q)}{N}}\right] \\
&= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\cdot N\mathbf{1}_{N\mathbb{Z}}(k-p)\cdot N\mathbf{1}_{N\mathbb{Z}}(l-q) \\
&= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\delta(k-p)\delta(l-q) = f(p,q).
\end{aligned}$$

(b) The matrix U used to calculate the DFT of an $N \times N$ matrix is given by

$$U = (U(x,\alpha))_{0\le x,\alpha\le n}, \text{ where } U(x,\alpha) = \frac{1}{\sqrt{N}}e^{2\pi j\frac{\alpha x}{N}}.$$

(c) Denote by $\vec{u}_\alpha$ the column of U indexed by α. Then for any $0 \le \alpha \le N-1$,

$$\begin{aligned}
\langle \vec{u}_\alpha, \vec{u}_\alpha\rangle &= \sum_{x=0}^{N-1} U(x,\alpha)\overline{U(x,\alpha)} \\
&= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}}e^{2\pi j\frac{x\alpha}{N}}\cdot\frac{1}{\sqrt{N}}e^{-2\pi j\frac{x\alpha}{N}} \\
&= N\cdot\frac{1}{N} = 1.
\end{aligned}$$

On the other hand, for any $0 \le \alpha_1, \alpha_2 \le N_1$ such that $\alpha_1 \ne \alpha_2$,

$$\begin{aligned}
\langle \vec{u}_{\alpha_1}, \vec{u}_{\alpha_2}\rangle &= \sum_{x=0}^{N-1} U(x,\alpha_1)\overline{U(x,\alpha_2)} \\
&= \sum_{x=0}^{N-1} \frac{1}{\sqrt{N}}e^{2\pi j\frac{x\alpha_1}{N}}\cdot\frac{1}{\sqrt{N}}e^{-2\pi j\frac{x\alpha_2}{N}} \\
&= \frac{1}{N}\sum_{x=0}^{N-1} e^{2\pi j\frac{x(\alpha_1-\alpha_2)}{N}} \\
&= \frac{1}{N}\cdot N\mathbf{1}_{N\mathbb{Z}}(\alpha_1-\alpha_2) = 0.
\end{aligned}$$

Hence U is unitary.

2. There are two different approaches.

Direct proof

$$\widehat{f\odot g}(k,l) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)g(m,n)e^{-2\pi j(\frac{mk}{M}+\frac{nl}{N})}.$$

On the other hand,

$$\begin{aligned}
\hat{f} * \hat{g}(k,l) &= \sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1} \hat{f}(a,b)\hat{g}(k-k',l-l') \\
&= \frac{1}{M^2N^2}\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1}\left(\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-2\pi j(\frac{mk'}{M}+\frac{nl'}{N})}\right) \\
&\left(\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} g(m',n')e^{-2\pi j(\frac{m'(k-k')}{M}+\frac{n'(l-l')}{N})}\right) \\
&= \frac{1}{M^2N^2}\sum_{k',m,m'=0}^{M-1}\sum_{l',n,n'=0}^{N-1} f(m,n)g(m',n')e^{-2\pi j(\frac{k'(m-m')+m'k}{M}+\frac{l'(n-n')+n'l}{N})} \\
&= \frac{1}{M^2N^2}\sum_{m,m'=0}^{M-1}\sum_{n,n'=0}^{N-1} f(m,n)g(m',n')e^{-2\pi j(\frac{m'k}{M}+\frac{n'l}{N})}M\delta(m-m')N\delta(n-n') \\
&= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)g(m,n)e^{-2\pi j(\frac{mk}{M}+\frac{nl}{N})} = \widehat{f\odot g}(k,l).
\end{aligned}$$

<u>Indirect proof (via inverse transform)</u>

$$\begin{aligned}
iDFT(\hat{f} * \hat{g})(m,n) &= \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} \hat{f} * \hat{g}(k,l)e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \sum_{k,k'=0}^{M-1}\sum_{l,l'=0}^{N-1} \hat{f}(k',l')\hat{g}(k-k',l-l')e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1}\sum_{k''=k-M+1}^{k}\sum_{l''=l-N+1}^{l} \hat{f}(k',l')\hat{g}(k'',l'')e^{2\pi j(\frac{m(k'+k'')}{M}+\frac{n(l'+l'')}{N})} \\
&= \left[\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1} \hat{f}(k',l')e^{2\pi j(\frac{mk'}{M}+\frac{nl'}{N})}\right]\left[\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1} \hat{g}(k'',l'')e^{2\pi j(\frac{mk''}{M}+\frac{nl''}{N})}\right] \\
&= f(m,n)g(m,n).
\end{aligned}$$

3. (a) The matrix T_{ec} used to calculate the EDCT of an $N \times N$ image is given by

$$T_{ec} = (T_{ec}(x,\alpha))_{0\le x,\alpha\le N-1}, \text{ where } T_{ec}(x,\alpha) = \frac{1}{N}\cos\frac{\pi x(\alpha+\frac{1}{2})}{N}.$$

(b) Let $N > 1$. Denote by $\vec{T}_x$ the row of T_{ec} indexed by x.
Suppose cT_{ec} is unitary for some $c \in \mathbb{R}$.
Then

$$1 = \langle c\vec{T}_0, c\vec{T}_0\rangle = \frac{c^2}{N^2}\sum_{\alpha=0}^{N-1}\cos^2\frac{\pi\cdot 0(\alpha+\frac{1}{2})}{N} = \frac{c^2}{N},$$

and thus $c^2 = N$.

On the other hand,

$$\begin{aligned}
1 &= \langle c\vec{T}_1, c\vec{T}_1\rangle \\
&= \frac{c^2}{N^2}\sum_{\alpha=0}^{N-1}\cos^2\frac{\pi\cdot 1(\alpha+\frac{1}{2})}{N} \\
&= \frac{c^2}{2N^2}\sum_{\alpha=0}^{N-1}[1+\cos(2\pi\frac{2\alpha+1}{2N})] \\
&= \frac{c^2}{2N}+\frac{c^2}{4N^2}\sum_{\alpha=-N}^{N-1}\cos(2\pi\frac{1\cdot(2\alpha+1)}{2N}) \\
&= \frac{c^2}{2N}+\frac{c^2}{2N}\mathbf{1}_{2N\mathbb{Z}}(1)=\frac{c^2}{2N}=\frac{1}{2}.
\end{aligned}$$

Contradiction. Hence cT_{ec} is not unitary for any $c\in\mathbb{R}$.

4. The given information implies

$$H(0,-1)=\frac{25}{26}\text{ and }H(-2,1)=\frac{1}{2}.$$

Recall that $H(u,v)=\dfrac{1}{1+(D(u,v)/D_0^2)^n}$, hence

$$\begin{cases}\dfrac{(D_0^2)^n}{(D_0^2)^n+1^n} &= \dfrac{25}{26}, \\ \dfrac{(D_0^2)^n}{(D_0^2)^n+5^n} &= \dfrac{1}{2},\end{cases}\text{ and thus }\begin{cases}25((D_0^2)^n+1) &= 26(D_0^2)^n, \\ (D_0^2)^n+5^n &= 2(D_0^2)^n.\end{cases}$$

Then $(D_0^2)^n=25=5^n$. Hence $n=2$ and $D_0^2=5$.

5. The given information implies

$$5(1-e^{-\frac{17}{2\sigma^2}})=4(1-e^{-\frac{34}{2\sigma^2}}),$$

which is a quadratic equation in $e^{-\frac{5}{\sigma^2}}$:

$$4(e^{-\frac{17}{2\sigma^2}})^2-5e^{-\frac{17}{2\sigma^2}}+1=0.$$

Hence $e^{-\frac{17}{2\sigma^2}}=\frac{1}{4}$ or 1 (rejected since $H<1$ for any $u,v\in\mathbb{R}$), and thus $-\frac{17}{2\sigma^2}=-\ln 4$ and $\sigma^2=\frac{17}{4\ln 2}$.

6. Note that $\mathcal{G}(f)=h*f$, where

$$h(x,y)=\begin{cases}-4 & \text{if } (x,y)=(0,0) \\ 1 & \text{if } x^2+y^2=1 \\ 0 & \text{otherwise.}\end{cases}$$

Then $DFT(\Delta f)=N^2DFT(h)\odot DFT(f)$, and thus

$$\begin{aligned}
H(u,v) &= N^2DFT(h)(u,v) \\
&= \sum_{x=0}^{N-1}\sum_{y=0}^{N-1}h(x,y)e^{-2\pi j\frac{ux+vy}{N}} \\
&= -4+(e^{2\pi j\frac{u}{N}}+e^{-2\pi j\frac{u}{N}}+e^{2\pi j\frac{v}{N}}+e^{-2\pi j\frac{v}{N}}) \\
&= -4+2\cos\frac{2\pi u}{N}+2\cos\frac{2\pi v}{N} \\
&= -4\sin^2\frac{\pi u}{N}-4\sin^2\frac{\pi v}{N}.
\end{aligned}$$

7. Note that $g = h * f$, where

$$h(i,j) = \begin{cases} \frac{1}{\lambda} & \text{if } i \leq \lambda - 1, j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $0 \leq u, v \leq N-1$. Then

$$\begin{aligned} DFT(h)(u,v) &= \frac{1}{\lambda N^2} \sum_{k=0}^{\lambda-1} e^{-2\pi j \frac{ku}{N}} \\ &= \begin{cases} \frac{1}{\lambda N^2} \frac{1-e^{-2\pi j \frac{\lambda u}{N}}}{1-e^{-2\pi j \frac{u}{N}}} & \text{if } e^{-2\pi j \frac{u}{N}} \neq 1 \\ \frac{1}{N^2} & \text{if } e^{-2\pi j \frac{u}{N}} = 1 \end{cases} \\ &= \begin{cases} \frac{1}{\lambda N^2} \frac{e^{-\pi j \frac{\lambda u}{N}} (e^{\pi j \frac{\lambda u}{N}} - e^{-\pi j \frac{\lambda u}{N}})}{e^{-\pi j \frac{u}{N}} (e^{\pi j \frac{u}{N}} - e^{-\pi j \frac{u}{N}})} & \text{if } u \notin N\mathbb{Z} \\ \frac{1}{N^2} & \text{if } u \in N\mathbb{Z} \end{cases} \\ &= \begin{cases} \frac{1}{\lambda N^2} e^{-\pi j \frac{(\lambda-1)u}{N}} \frac{\sin \frac{\lambda \pi u}{N}}{\sin \frac{\pi u}{N}} & \text{if } u \neq 0 \\ \frac{1}{N^2} & \text{if } u \neq 0. \end{cases} \end{aligned}$$

Since $DFT(h * f) = N^2 DFT(h) \odot DFT(f)$,

$$H(u,v) = N^2 DFT(h)(u,v) = \begin{cases} \frac{1}{\lambda} e^{-\pi j \frac{(\lambda-1)u}{N}} \frac{\sin \frac{\lambda \pi u}{N}}{\sin \frac{\pi u}{N}} & \text{if } u \neq 0 \\ 1 & \text{if } u = 0 \end{cases}.$$