MMAT5390: Mathematical Imaging

Assignment 4

Due: 7 May 2021

Please give detailed steps and reasons in your solutions.

1. Given a noisy image $g(x,y)$, we consider the image denoising algorithm to obtain a clean image $f(x,y)$ through minimizing the following energy functional:

$$E(f) = \int_{\Omega} \{|f(x,y) - g(x,y)| + \lambda \|\nabla f(x,y)\|^2\} \, dx \, dy$$

where λ is a constant parameter.

(a) Discretize $E(f)$.

(b) Find a non-linear equation that $f(x,y)$ must satisfy at every position $(x,y) \in S$ in order to minimize the discrete version of $E(f)$, where $S = \{(x,y) \in \mathbb{Z}^2 : (x,y), (x \pm 1, y), (x, y \pm 1) \in \Omega\}$. You may need to introduce a small parameter $\varepsilon > 0$ to avoid singularities.

(c) Design an iterative scheme to minimize the discrete version of $E(f)$.

(d) Now, given a noisy image $g(x,y)$, we consider the image denoising algorithm to obtain a clean image $f(x,y)$ through minimizing the following energy functional:

$$E(f) = \int_{\Omega} \{|f(x,y) - g(x,y)|^2 + \lambda \|\nabla f(x,y)\|^4\} \, dx \, dy$$

where λ is a constant parameter. Derive an iterative scheme to minimize $E(f)$ in the continuous setting.

2. Consider the following curve evolution model for image segmentation. Let $\gamma_t := \gamma_t(s) : [0, 2\pi] \to D$ be a closed curve in the image domain D. We proceed to find γ that minimizes:

$$E_{snake,2}(\gamma) = \int_0^{2\pi} \frac{1}{2} \|\gamma'(s)\|^2 \, ds + \alpha \int_0^{2\pi} \frac{1}{2} \|\gamma''(s)\|^2 \, ds + \beta \int_0^{2\pi} V(\gamma(s)) \, ds,$$

where V is the edge detector, α and β are fixed positive parameters. Assume that γ' and γ'' are discretized by:

$$\gamma'(s_i) = [\gamma(s_{i+1}) - \gamma(s_i)]/\sigma \text{ and}$$
$$\gamma''(s_i) = [\gamma(s_{i+1}) - 2\gamma(s_i) + \gamma(s_{i-1})]/\sigma^2.$$

(a) Derive the gradient descent iterative scheme to minimize $E_{snake,2}$ in the continuous setting.

(b) Discretize $E_{snake,2}$.

(c) Derive the explicit Euler scheme (using gradient descent method) to iteratively minimize the discrete version of $E_{snake,2}$.

3. Derive the anisotropic diffusion equation:

$$\frac{\partial I(x, y; \sigma)}{\partial \sigma} = \nabla \cdot (K(x, y) \nabla I(x, y; \sigma))$$

from minimizing the following energy functional:

$$E(I) = \int_{\Omega} K(x, y) \|\nabla I(x, y)\|^2 \, dx \, dy.$$