MMAT5010 2021 Home Test 2

Q1. (a) Let $x_0 \in X$ be non-zero. By Hahn-Banach Theorem, there exists $x^* \in X^*$ with $||x^*|| = 1$ and $x^*(x_0) = ||x_0||$. Then we can define $f : X \to \mathbb{R}$ to be $f(x) = ||x_0||^{-1}x^*(x)$.

(b) Let f by (a). We can define $T : X \to X$ to be $T(x) = f(x)x_0$. Then $||T|| = ||f||\,||x_0|| = 1$ and $Tx_0 = x_0$.

Q2. (a) Note that the natural basis vectors of ℓ_p, e_i, $i = 1, 2, \dots$,lie in X. But $Te_n = ne_n$. Hence T is unbounded.

(b) In this case T must not be an isomorphism because it is not even continuous.

Q3. (b) If (x_n) is weakly convergent to both $x, y \in H$, then $x = y$ must hold. Otherwise, by Hahn Banach Theorem there is $f \in H^*$ such that $f(x) \neq f(y)$. But $\lim_n f(x_n) = f(x) = f(y)$ by assumption.

(b) We only need to show that forward direction. Note

$$||x_n - x||^2 = ||x_n||^2 - \langle x_n, x\rangle - \langle x, x_n\rangle + ||x||^2$$

By weak convergence, $\langle x_n, x\rangle \to 0$. Hence $||x_n - x||^2 \to 0$.

Q4. (a) Suppose $P-Q$ is an orthogonal projection. Let $x \in N$. Then $\langle x-(P-Q)x, (P-Q)x\rangle = 0$, i.e. $\langle x, Px - x\rangle - ||Px - x||^2 = 0$. Note $\langle x, Px - x\rangle = -||Px - x||^2 + \langle Px, Px - x\rangle$. Because $Px - x \in M^\perp$ and $Px \in M$, combining everything gives $||Px - x|| = 0$. Hence $x \in M$.

Suppose $N \subset M$. Let $x \in H$. Then

$$\begin{aligned}
\langle x - (P - Q)x, (P - Q)x\rangle &= \langle x - P(x - Qx), P(x - Qx)\rangle \\
&= \langle x - Qx + Qx - P(x - Qx), P(x - Qx)\rangle \\
&= \langle Qx, Px - Qx\rangle \\
&= \langle Qx, QPx - Qx\rangle + \langle Qx, Px - QPx\rangle
\end{aligned}$$

where in the above $Px = QPx + Px - QPx$ is in its orthogonal decomposition, so that $Px - QPx \in N^\perp$, and the second term $= 0$. For the first term, notice that $Px - x \in M^\perp \subset N^\perp$, so $QPx - Qx = 0$, so the first term $= 0$ as well. Hence $P - Q$ is an orthogonal projection.

(b) Suppose that $P - Q$ is an orthogonal projection, then $N \subset M$.

Claim. $(P - Q)(H) \subset M \cap N^\perp$.

Suppose $y \in (P - Q)(H)$. Note $y \in N^\perp$ if and only if $Qy = 0$. We know that there is $x \in H$ such that $Px - Qx = y$. Therefore $Qy = QPx - Qx = 0$. (here $QPx = Qx$)

Claim. $M \cap N^\perp \subset (P - Q)(H)$.

If $x \in M \cap N^\perp$. First $x \in M$ implies $Px = x$, $x \in N^\perp$ implies $Qx = 0$. Hence $Px - Qx = x$, $x \in (P - Q)(H)$.