MATH 5061 Problem Set 1[1]
Due date: Jan 27, 2021

Problems: (Please hand in your assignments via Blackboard. **Late submissions will not be accepted.**)

1. Let M be the Möbius band defined as the topological quotient of $[0,1] \times \mathbb{R}$ by the equivalence relation $(0,t) \sim (1,-t)$ for any $t \in \mathbb{R}$.
   (a) Show that M can be equipped with a differentiable structure which is consistent with the topology.
   (b) Prove that M is not orientable.
   (c) Show that $\mathbb{RP}^2$ can be obtained by gluing together a disk with a Möbius band along their boundary. Use this to show that $\mathbb{RP}^2$ is not orientable.
2. Construct an explicit diffeomorphism between $\mathbb{S}^2$ and $\mathbb{CP}^1$.
3. Compute the tangent space of $SO(n)$ at the identity matrix I, and use this to compute the dimension of $SO(n)$ as a manifold. What is the tangent space of $SO(n)$ at an arbitrary $A \in SO(n)$.
4. Prove that the tangent bundle TM is always orientable as a manifold.
5. (a) Show that a rank n vector bundle $\pi : E \to B$ is trivial if and only if there exist n linearly independent sections $\{s_i\}_{1 \leq i \leq n}$, i.e. at every point $b \in B$, $\{s_i(b)\}_{1 \leq i \leq n}$ forms a linearly independent set of the vector space $\pi^{-1}(b)$.
   (b) Show that the Möbius band as defined in Problem 1 is the total space of a non-trivial vector bundle of rank 1 over S^1.

[1]Last revised on January 19, 2021