Lecture 16

March 11, 2021

1 Convergence theorems

We are going to prove the pointwise convergence of the classical full Fourier series.

For a C^1 function $f(x)$ on $(-\pi, \pi)$ the Fourier series is

$$S(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B \sin nx)$$

with the coefficients

$$A_n = \int_{-\pi}^{\pi} f(y) \cos ny \frac{dy}{\pi} \qquad (n = 0, 1, 2, \cdots)$$

$$B_n = \int_{-\pi}^{\pi} f(y) \sin ny \frac{dy}{\pi} \qquad (n = 1, 2, \cdots)$$

The Nth partial sum of the series is

$$S_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx).$$

We want to prove that $S_N(x)$ converges to $f(x)$ as $N \to \infty$. So the Fouries series $S(x)$ equals the function $f(x)$ in $(-\pi, \pi)$. Replacing the formulas A_n and B_n into $S_N(x)$, we have

$$\begin{aligned} S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[1 + 2\sum_{n=1}^{N}(\cos ny \cos nx + \sin ny \sin nx)]f(y)dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[1 + 2\sum_{n=1}^{N}\cos n(x-y)]f(y)dy. \end{aligned} \qquad (1)$$

Denote the *Dirichlet kernel* K_N to be

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N}\cos n\theta. \qquad (2)$$

Because of the observation

$$2\cos n\theta \sin\frac{1}{2}\theta \quad = \quad \sin(n+\frac{1}{2})\theta - \sin(n-\frac{1}{2})\theta.$$

Thus we have

$$\begin{aligned} K_N(\theta) &= 1 + \sum_{n=1}^{N} \frac{\sin(n+\frac{1}{2})\theta - \sin(n-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} \\ &= 1 + \frac{\sin(N+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} \\ &= \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}. \end{aligned}$$

The graph of the Dirichlet kernel K_N is

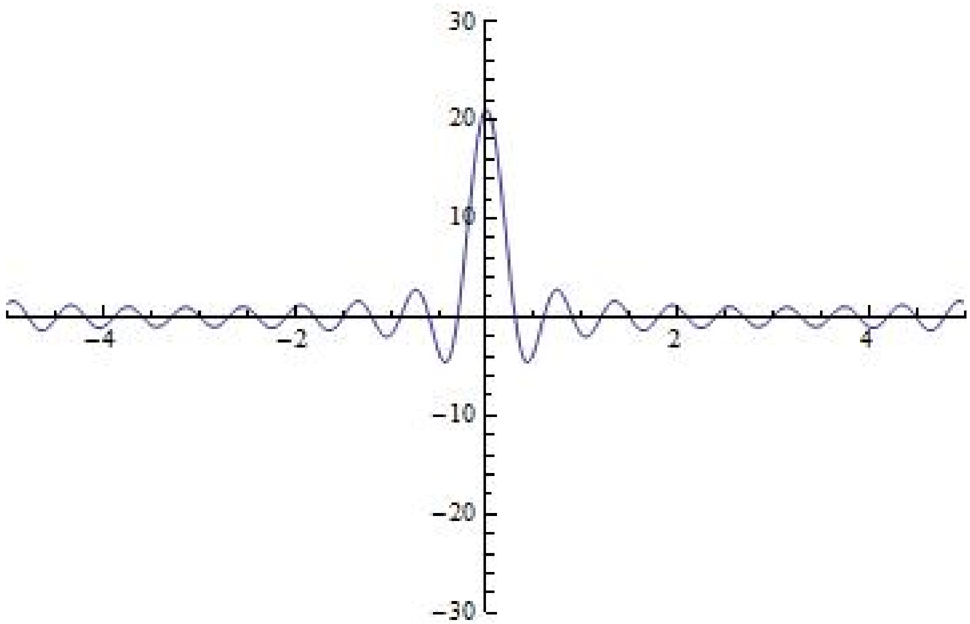

Figure 1: $N = 10$

Compare this with the heat kernel $S_t(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$. The graph for S_t is

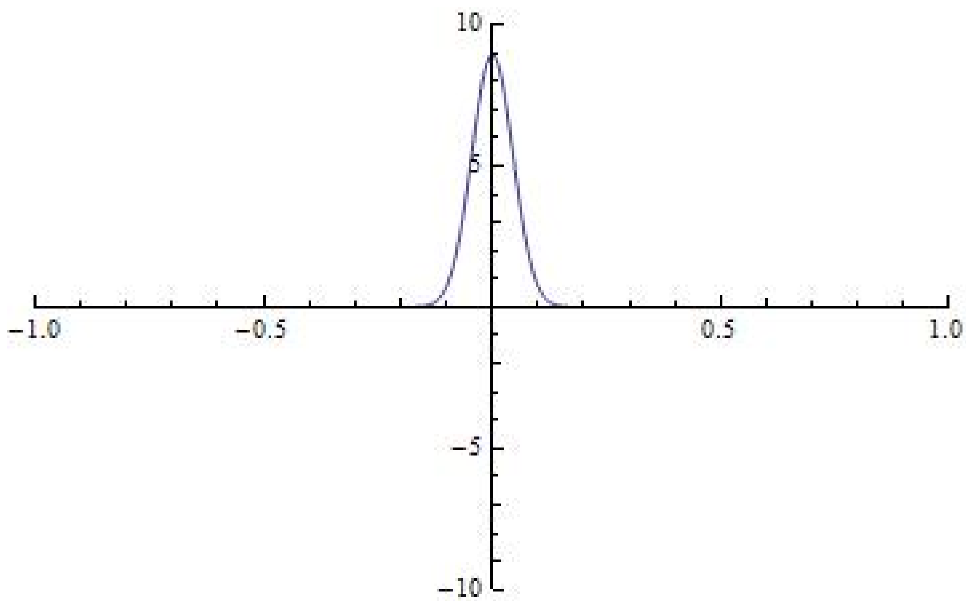

Figure 2: $t = 0.001$

Letting $\theta = y - x$ and using the evenness of K_N, formula (1) takes the form

$$S_N(x) \quad = \quad \int_{-\pi}^{\pi} K_N(y-x) f(y) \frac{dy}{2\pi}.$$

Notice that by the definition of K_N

$$\begin{aligned} \int_{-\pi}^{\pi} K_N(y-x) \frac{dy}{2\pi} &= \int_{-\pi}^{\pi} [1 + 2\sum_{n=1}^{N} \cos n(y-x)] \frac{dy}{2\pi} \\ &= 1. \end{aligned}$$

Then

$$\begin{aligned} S_N(x) - f(x) &= \int_{-\pi}^{\pi} K_N(y-x)[f(y) - f(x)] \frac{dy}{2\pi} \\ &= \int_{-\pi}^{\pi} \sin(N + \frac{1}{2})(y-x) \frac{[f(y) - f(x)]}{2\sin\frac{(y-x)}{2}} \frac{dy}{\pi}. \end{aligned}$$

We have assumed that $f(x)$ has a differentiable derivative, so $\frac{f(x)-f(y)}{x-y}$ and $h(\theta) = \frac{f(y)-f(x)}{x-y} \frac{x-y}{2\sin\frac{(y-x)}{2}}$ are continuous functions with respect the variable θ.

Then

$$\begin{aligned} S_N(x) - f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(N+\frac{1}{2})(y-x)h(y-x)dy \\ &= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} \sin(N+\frac{1}{2})\theta h(\theta)d\theta. \end{aligned}$$

Because $\{X_n(\theta)\}$=$\{\sin(n+\frac{1}{2})\theta\}$ are an orthogonal set of functions on the interval $(-x, \pi-x)$. Hence they are also orthogonal on the interval $(-x-\pi, -x+\pi)$. Due to the least-Square Approximation theorem, if $\|h\|^2 = (h,h) < \infty$ we have from Bessel's inequality

$$\sum_{n=1}^{\infty} \frac{(h, X_n)^2}{(X_n, X_n)} \leq \|h\|^2.$$

By direct calculation

$$(X_n, X_n) = \int_{-\pi-x}^{\pi-x} \sin^2(N+\frac{1}{2})\theta d\theta = \pi.$$

So we have for bigger N,

$$(h, X_N) \to 0 \quad as \quad N \to \infty.$$

Then we check that $\|h\|^2 < \infty$ which is

$$\int_{-\pi-x}^{\pi-x} h^2(\theta)d\theta = \int_{-\pi-x}^{\pi-x} [\frac{f(x+\theta)-f(x)}{2\sin\frac{\theta}{2}}]^2 d\theta < \infty.$$

The above inequality is true because h is a continues function.

Exercise 1. If a period function $f(x)$ itself is only piecewise continuous and $f'(x)$ is also piecewise continuous on $-\infty < x < \infty$, prove that for any fixed x

$$\lim_{N\to\infty} |S_N(x) - \frac{1}{2}[f(x+) + f(x-)]| = 0.$$

We are going to prove the *uniform convergence* theorem for classical Fourier series. We assume again that $f(x)$ and $f'(x)$ are continuous functions of period of 2π.

Denote A_n and B_n are the Fourier coefficients of $f(x)$ and let A'_n and B'_n are the Fourier coefficients of $f'(x)$.

We integrate by parts to get

$$\begin{aligned} A_n &= \int_{-\pi}^{\pi} f(x)\cos nx \frac{dx}{\pi} \\ &= \frac{1}{n\pi} f(x)\sin nx|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\sin nx\frac{dx}{n\pi} \\ &= -\int_{-\pi}^{\pi} f'(x)\sin nx \frac{dx}{n\pi} \\ &= -B'_n. \end{aligned}$$

Similarly,

$$B_n = -\frac{1}{n}A'_n.$$

Due to Bessel's inequality for the functions $f'(x)$ that the infinite series

$$\sum_{n=1}^{\infty}(|A'_n|^2 + |B'_n|^2) \le \pi \int_{-\pi}^{\pi} |f'(x)|^2 dx < \infty.$$

$$\begin{aligned}
\frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B \sin nx) &\le \frac{1}{2}|A_0| + \sum_{n=1}^{\infty}(|A_n| + |B_n|) \\
&\le \frac{1}{2}|A_0| + \sum_{n=1}^{\infty}\frac{1}{n}(|A'_n| + |B'_n|) \\
&\le \frac{1}{2}|A_0| + (\sum_{n=1}^{\infty}\frac{1}{n^2})^{\frac{1}{2}}[\sum_{n=1}^{\infty}(|A'_n| + |B'_n|)^2]^{\frac{1}{2}} \\
&\le \frac{1}{2}|A_0| + (\sum_{n=1}^{\infty}\frac{1}{n^2})^{\frac{1}{2}}[2\sum_{n=1}^{\infty}(|A'_n|^2 + |B'_n|^2)]^{\frac{1}{2}} \\
&< \infty.
\end{aligned}$$

Here we used the Schwarz inequality for infinite series:

$$\sum_{n=1}^{\infty} a_n b_n \le (\sum_{n=1}^{\infty} a_n^2)^{\frac{1}{2}} (\sum_{n=1}^{\infty} b_n^2)^{\frac{1}{2}}.$$

So the Fourier series converges *absolutely*.
Moreover, we have

$$\begin{aligned}
\max|f(x) - S_N(x)| &\le \sum_{n=N+1}^{\infty} |A_n \cos nx + B_n \sin nx| \\
&\le \sum_{n=N+1}^{\infty} (|A_n| + |B_n|) \\
(asN \to \infty) &\to 0.
\end{aligned}$$

2 The Gibbs phenomenon

Let $f(x)$ be a step function with a jump

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0. \end{cases}$$

Note from the previous discussion, we have

$$\lim_{N\to\infty} |S_N(0) - \frac{1}{2}[f(0+) + f(0-)]| \quad = \quad 0.$$

In fact,

$$S_N(0) \quad = \quad \frac{1}{2\pi}[\int_0^\pi \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy - \int_{-\pi}^0 \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy]$$

Then

$$\begin{aligned} |S_N(0) - \frac{1}{2}[f(0+) + f(0-)]| &= \frac{1}{2\pi}\int_0^\pi [\frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} - 1]dy - \frac{1}{2\pi}\int_{-\pi}^0 [\frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} - 1]dy \\ &= \frac{1}{2\pi}\int_0^\pi 2\sum_{n=1}^N \cos ny dy - \frac{1}{2\pi}\int_{-\pi}^0 2\sum_{n=1}^N \cos ny dy \\ &= 0. \end{aligned}$$

But we are going to prove that for some $x_N \to 0$

$$\lim_{N\to\infty} S_N(x_N) \quad \neq \quad 0.$$

Moreover, this limit is 9 percent higher than the jump of the function f. Here the jump is 2.

This is called Gibbs phenomenon.

Let $x_N = \frac{\pi}{N+\frac{1}{2}}$, then the partial sum S_N is

$$\begin{aligned} S_N(x_N) &= \int_{-\pi}^\pi K_N(y - x_N) f(y) \frac{dy}{2\pi} \\ &= \frac{1}{2\pi}[\int_{-x_N}^{\pi - x_N} K_N(\theta) d\theta - \int_{-\pi-x_N}^{-x_N} K_N(\theta) d\theta] \\ &= \frac{1}{2\pi}[\int_{-x_N}^{\pi - x_N} \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} d\theta + \int_{\pi+x_N}^{x_N} \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} d\theta] \\ &= \frac{1}{2\pi}[\int_{\pi+x_N}^{\pi - x_N} \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} d\theta + \int_{-x_N}^{x_N} \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} d\theta]. \end{aligned}$$

We are going to estimate the above two integrals

$$\frac{1}{2\pi}\int_{\pi+x_N}^{\pi-x_N} \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} d\theta \to \frac{1}{2\pi}\int_\pi^\pi \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} d\theta \to 0 \quad as \quad N \to 0. \ (3)$$

And

$$\begin{aligned}
\frac{1}{2\pi}\int_{-x_N}^{x_N}\frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}d\theta &= \frac{1}{2\pi}\int_{-\frac{\pi}{N+\frac{1}{2}}}^{\frac{\pi}{N+\frac{1}{2}}}\frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}d\theta \\
(let\varphi=(N+\frac{1}{2})\theta) &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin\varphi}{(2N+1)\sin\frac{\varphi}{2N+1}}d\varphi \\
&\to \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin\varphi}{\varphi}d\varphi \\
&\approx 1.179. \qquad (4)
\end{aligned}$$

Combining (3) and (4), we have

$$\lim_{N\to\infty}S_N(x_N) \quad\approx\quad S_{20}(x_{20})\approx 1.179\approx 9\%*2+1.$$

This is Gibbs's 9 percent overshoot phenomenon.
The graph for $S_N(x)=(\int_0^{\pi}-\int_{-\pi}^{0})\frac{\sin[(N+\frac{1}{2})(x-y)]}{\sin\frac{1}{2}(x-y)}\frac{dy}{2\pi}$

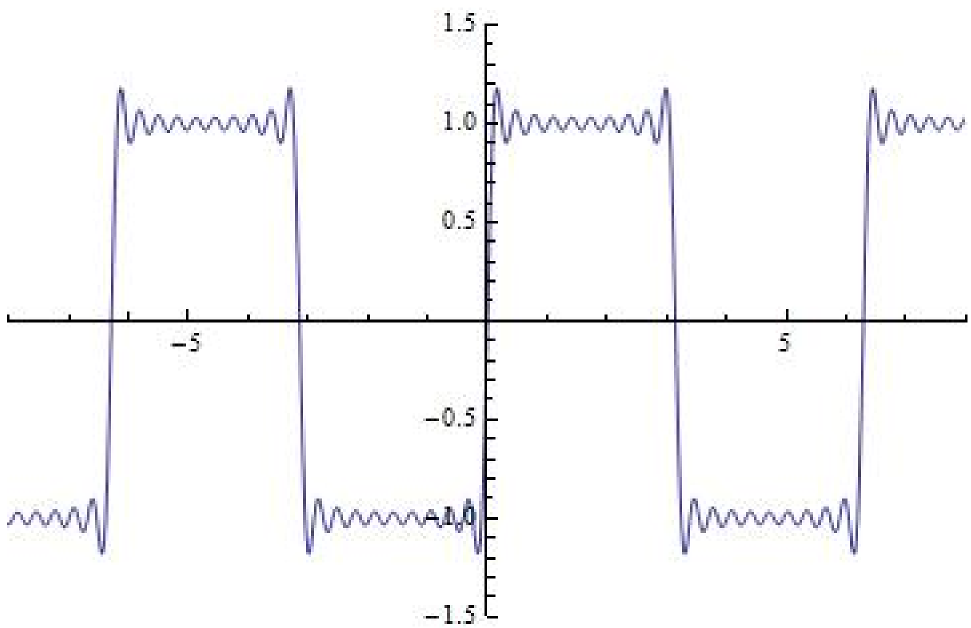

Figure 3: $N=20$